

# On a Diagonalization of Matrices over Regular Ring

Yoshito YUKIMOTO \*

**Abstract.** An element  $x$  in a ring  $R$  is called right (resp. left) invertible if there exists  $y \in R$  such that  $xy = 1$  (resp.  $yx = 1$ ). An element in a ring  $R$  is called invertible if it is right invertible and left invertible. In this note we show that, for any right invertible matrix  $X$  in the ring  $M_2(R)$  of all (2,2)-matrices over a regular ring  $R$ , there exist an element  $Y \in M_2(R)$  and an invertible matrix  $V \in M_2(R)$  such that  $XVY = I$  (identity matrix) and  $YXV$  is a diagonal matrix.

## 1. Introduction.

An element  $a$  in a ring  $R$  is said to be a generalized inverse of  $x \in R$  if  $x = xax$  and  $a = axa$ . A ring  $R$  is called (von Neumann) regular if for any element  $x$  in  $R$  there exists a generalized inverse of  $x$ . This generalized inverse of  $x$  is not uniquely determined in general. A ring  $R$  is called directly finite if  $xy = 1$  ( $x, y \in R$ ) implies  $yx = 1$ . The endomorphism ring of an  $n$ -dimensional vector space ( $n < \infty$ ), which is isomorphic to the ring of all matrices of degree  $n$ , is an example of directly finite regular ring, and the endomorphism ring  $S$  of an infinite dimensional vector space is a directly infinite regular ring. The matrix ring  $M_2(S)$  over this ring is also a directly infinite regular ring, and  $XY = I$  does not imply  $YX = I$ . But  $M_2(S)$  has some similarity to matrix ring  $M_2(F)$  over a field  $F$ . This similarity is stated in the following general form, which is proved in this note:

For any right invertible matrix  $X$  in the ring  $M_2(R)$  of all (2,2)-matrices over a regular ring  $R$ , there exist an element  $Y \in M_2(R)$  and an invertible matrix  $V \in M_2(R)$  such that  $XVY = I$  (identity matrix) and  $YXV$  is a diagonal matrix.

## 2. Transformations of matrix.

In this section all matrices are (2,2)-matrices over a regular ring. If  $M$  is right invertible element of a ring and  $U, V$  are invertible elements of the ring then  $UMV$  is right invertible. Especially the right invertibility is reserved under elementary transformations.

In the following we transform a right invertible matrix to a right invertible matrix of a special type.

Let  $\begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix}$  be a right invertible matrix,  $a$  a generalized inverse of  $x_1$ . Then

$$\begin{bmatrix} 1 & 0 \\ -z_1a & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ z_1(1 - ax_1) & -z_1ay_1 + w_1 \end{bmatrix}$$

and  $z_1(1 - ax_1)a = 0$ .

Let  $\begin{bmatrix} x_2 & y_2 \\ z_2 & w_2 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ z_1(1 - ax_1) & -z_1ay_1 + w_1 \end{bmatrix}$ ,  $b$  a generalized inverse of  $z_2$ . Then  $z_2a = 0$ .

$$\begin{bmatrix} 1 & -x_2b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 & y_2 \\ z_2 & w_2 \end{bmatrix} = \begin{bmatrix} x_2(1 - bz_2) & y_2 - x_2bw_2 \\ z_2 & w_2 \end{bmatrix},$$

$x_2(1 - bz_2)b = 0$  and  $a$  is a generalized inverse of  $x_2(1 - bz_2)$ .

---

\* Liberal Arts

Let  $\begin{bmatrix} x_3 & y_3 \\ z_3 & w_3 \end{bmatrix} = \begin{bmatrix} x_2(1-bz_2) & y_2 - x_2bw_2 \\ z_2 & w_2 \end{bmatrix}$  Then  $z_3a = 0, x_3b = 0$  and

$$\begin{bmatrix} x_3 & y_3 \\ z_3 & w_3 \end{bmatrix} \begin{bmatrix} 1 & -ay_3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x_3 & (1-x_3a)y_3 \\ z_3 & w_3 \end{bmatrix}$$

Since this matrix is right invertible,  $x_3aR \oplus (1-x_3a)y_3R = x_3R \oplus (1-x_3a)y_3R = R$  and there exist  $\alpha, \gamma \in R$  satisfying  $x_3a\alpha + (1-x_3a)y_3\gamma = 1$ . Multiplying  $a$  on the left of both sides of this equation, we have  $a\alpha = a$  and  $x_3a + (1-x_3a)y_3\gamma = 1$ . Moreover  $\gamma(1-x_3a)$  is a generalized inverse of  $(1-x_3a)y_3$ .

Let  $\begin{bmatrix} x_4 & y_4 \\ z_4 & w_4 \end{bmatrix} = \begin{bmatrix} x_3 & (1-x_3a)y_3 \\ z_3 & w_3 \end{bmatrix}$ ,  $c = \gamma(1-x_3a)$ . Then  $a, b, c$  are generalized inverses of  $x_4, z_4, y_4$  respectively,  $x_4b = 0, z_4a = 0, x_4a + y_4c = 1$ . By the last equation we have  $cx_4 = cx_4ax_4 = c(1-y_4c)x_4 = 0$  and  $ay_4 = ay_4cy_4 = a(1-ax_4)y_4 = 0$ . This matrix is transformed to

$$\begin{bmatrix} 1 & 0 \\ -w_4c & 1 \end{bmatrix} \begin{bmatrix} x_4 & y_4 \\ z_4 & w_4 \end{bmatrix} = \begin{bmatrix} x_4 & y_4 \\ z_4 & w_4(1-cy_4) \end{bmatrix}$$

Let  $\begin{bmatrix} x_5 & y_5 \\ z_5 & w_5 \end{bmatrix} = \begin{bmatrix} x_4 & y_4 \\ z_4 & w_4(1-cy_4) \end{bmatrix}$ . Then  $a, b, c$  are generalized inverses of  $x_5, z_5, y_5$  respectively,  $x_5a + y_5c = 1, x_5b = 0, z_5a = 0$  and  $w_5c = 0$ . Since the matrix

$$\begin{bmatrix} x_5 & y_5 \\ z_5 & w_5 \end{bmatrix} \begin{bmatrix} 1 & -bw_5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x_5 & y_5 \\ z_5 & (1-z_5b)w_5 \end{bmatrix}$$

is right invertible,  $z_5bR \oplus (1-z_5b)w_5R = z_5R \oplus (1-z_5b)w_5R = R$  and there exist  $\beta, \delta \in R$  satisfying  $z_5b\beta + (1-z_5b)w_5\delta = 1$ . Multiplying  $b$  on the left of both sides of this equation, we have  $b\beta = b$  and  $z_5b + (1-z_5b)w_5\delta = 1$ . Moreover  $\delta(1-z_5b)$  is a generalized inverse of  $(1-z_5b)w_5$ .

Let  $\begin{bmatrix} x_6 & y_6 \\ z_6 & w_6 \end{bmatrix} = \begin{bmatrix} x_5 & y_5 \\ z_5 & (1-z_5b)w_5 \end{bmatrix}$  and  $d = \delta(1-z_5b)$ . Then  $a, b, c, d$  are generalized inverses of  $x_6, z_6, y_6, w_6$  respectively,  $x_6a + y_6c = 1, z_6b + w_6d = 1, x_6b = 0, z_6a = 0$  and  $w_6c = 0$ . By the two equations  $x_6a + y_6c = 1, z_6b + w_6d = 1$  we have  $ay_6 = 0, cx_6 = 0, bw_6 = 0, dz_6 = 0$ , and

$$\begin{bmatrix} 1 & -y_6d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_6 & y_6 \\ z_6 & w_6 \end{bmatrix} = \begin{bmatrix} x_6 & y_6(1-dw_6) \\ z_6 & w_6 \end{bmatrix}$$

Since  $y_6(1-dw_6)c = yc$ ,  $c$  is also a generalized inverse of  $y_6(1-dw_6)$  and  $x_6a + y_6(1-dw_6)c = 1$ .

Let  $\begin{bmatrix} x_7 & y_7 \\ z_7 & w_7 \end{bmatrix} = \begin{bmatrix} x_6 & y_6(1-dw_6) \\ z_6 & w_6 \end{bmatrix}$ . Then  $a, b, c, d$  are generalized inverses of  $x_7, z_7, y_7$ , and  $x_7b = 0, z_7a = 0, y_7d = 0, w_7c = 0$ .

### 3. Canonical form.

In the previous section we show that any right invertible matrix  $\begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix}$  is transformed to a matrix  $\begin{bmatrix} x_7 & y_7 \\ z_7 & w_7 \end{bmatrix}$ , and there exist generalized inverses  $a, b, c, d$  of  $x_7, z_7, y_7, w_7$  (resp.) such that  $x_7a + y_7c = 1, z_7b + w_7d = 1$  and  $x_7b = 0, z_7a = 0, y_7d = 0, w_7c = 0$ . Hence

$$\begin{bmatrix} x_7 & y_7 \\ z_7 & w_7 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_7 & y_7 \\ z_7 & w_7 \end{bmatrix}$$

is a diagonal matrix.

Briefly speaking, for any right invertible (2,2)-matrix  $X$ , there exist two invertible (2,2)-matrices  $U, V$  and a (2,2)-matrix  $Y_1$  such that  $UXVY_1 = I$  and  $Y_1UXV$  is a diagonal matrix. By  $UXVY_1 = I$  we have  $XVY_1 = U^{-1}$  and  $XVY_1U = I$ . By setting that  $Y = Y_1U$ , we have the following theorem.

**THEOREM.** *For any right invertible (2,2)-matrix  $X$ , there exist an (2,2)-matrix  $Y$  and an invertible (2,2)-matrix  $V$  such that  $XVY = I$  and  $YXV$  is a diagonal matrix.*

### REFERENCE

1. K.R. Goodearl, "Von Neumann Regular Rings," Pitman, London, 1979.

(Received December 17, 1993)