ある距離化定理の別証について

石 川 文 江

Another Proofs of Some Metrization Theorems

Fumie ISHIKAWA

Abstract

The purpose of this paper is to give another proofs of certain well known metrization theorems by the method of ranked space.

§0. Introduction

K. Kunugi introduced the notion of ranked space as a generalization of metric spaces (see〔3〕). There are various theorems which have been established in the problem of finding necessary and sufficient conditions for a topological space to be homeomorphic with metric space. In this note we shall prove some metrization theorems in the method of ranked space. Throughout this note, the term "ranked space" will means a ranked space of indicator ω_0. (ω_0 is the first nonfinite ordinal)

§1. Preliminaries

We define the ranked space. Let R be a nonempty set such that, to every point p of R, there correspond a non-empty family $\mathfrak{V}(p)$ whose elements are subsets of R, denoted by $V(p)$, $\mathfrak{U}(p)$, etc. and called preneighborhoods of p. Suppose that, for every p of R and every preneighborhood $V(p)$ in $\mathfrak{V}(p)$ satisfy the following condition:

(A) (Axiom (A) of Hausdorff see〔2〕). $V(p)\ni p$.

Define $\mathfrak{V}=\cup\{\mathfrak{V}(p);\ p \text{ in } R\}$. Then the space R is called to be a ranked space if for every $n\in N$ (N is the set of $\{0, 1, 2, \cdots\cdots\}$), there is associated a subfamily of $\mathfrak{V}$, denoted by $\mathfrak{V}_n$, satisfing the following axiom:

(a) For every $p\in R$, every $V(p)\in\mathfrak{V}(p)$ and every $n\in N$, we can find a $U(p)$ such that

(1) $U(p)\subset V(p)$

(2) $U(p)$ belongs to some $\mathfrak{V}_m$ with $m\geqq n$.

A preneighborhood belonging to $\mathfrak{V}_n$ is said to be with rank n. Preneighborhoods of p with rank n are written $V(p, n)$, $U(p, n)$, etc.. Moreover we assume that R is a preneighborhood of every point with rank 0. A ranked space is a nonempty set R with those families $\mathcal{V}$, $\mathcal{V}_n(n \in N)$, which is written $(R, \mathcal{V}, \mathcal{V}_n)$ (briefly, $(R, \mathcal{V})$). In a ranked space $(R, \mathcal{V})$ a sequence of preneighborhoods $\{V_i(p_i, n_i)\}$ (briefly, $\{V_i\}$) is called a fundamental (or more precisely $\mathcal{V}$-fundamental) sequence if three conditions belw are fulfilled.

(1) $V_0(p_0, n_0) \supset V_1(p_1, n_1) \supset \cdots\cdots \supset V_i(p_i, n_i) \supset \cdots\cdots$,

(2) $n_0 \leqq n_1 \leqq \cdots\cdots \leqq n_i \leqq \cdots\cdots$, $0 \leqq n_i < \infty$, $\lim_{i \to \infty} n_i = \infty$

(3) For every $n \in N$, there exists an $i \in N$ such that $i \geqq n$, $p_i = p_{i+1}$ and $n_i < n_{i+1}$.

In paticular, $\{V_i(p_i, n_i)\}$ is called fundamental sequence of center p, if $p_i = p$ for all i. A sequence $\{p_i\}$ in R is called a Cauchy sequence if there exists a fundamental sequence of preneighborhoods $\{V_i(q_i, n_i)\}$ such that for every V_i there exists a j with the property that $p_k \in V_i$ for all $k \geqq j$. In this case, $\{V_i\}$ is called a defining sequence of the Cauchy sequence $\{p_i\}$. A sequence $\{p_i\}$ in R is said to ortho- (or r-) [resp. para- (or π-)] convergence to p if $\{p_i\}$ is a Cauchy sequence for which we can find a defining sequence $\{V_i(p, n_i)\}$ [resp. $\{V_i(q_i, n_i)\}$ such that $p \in \bigcap_{i \in N} V_i(q_i, n_i)$].

A ranked space is said to be complete, if for every fundamental sequence $\{V_i\}$ we have $\bigcap_{i \in N} V_i \neq \phi$. For two fundamental sequences $\{V_i\}$ and $\{U_i\}$ we write $\{V_i\} > \{U_i\}$ to mean that for every V_i, there exists a U_j such that $V_i \supset U_j$ and $\{V_i\}$ and $\{U_i\}$ are said to be equivalent if $\{V_i\} > \{U_i\}$ and $\{V_i\} < \{U_i\}$.

Two ranked space $(R, \mathcal{V})$ and $(R, \mathcal{U})$ are said to be equivalent (with respect to fundamental sequence) if for every $\mathcal{V}$-fundamental sequence $\{V_i(p, n_i)\}$ [resp. $\{V_i(q_i, n_i)\}$] there exists an equivalent $\mathcal{U}$-fundamental sequence $\{U_i(p, n_i)\}$ [resp. $\{U_i(r_i, m_i)\}$] and for every $\mathcal{U}$-fundamental sequence $\{U_i(p, n_i)\}$ [resp. $\{U_i(q_i, n_i)\}$] there exists an equivalent $\mathcal{V}$-fundamental sequence $\{V_i(p, m_i)\}$ [resp. $\{V_i(r_i, m_i)\}$].

§ 2. Metrization of ranked spaces

A ranked space satisfies the axiom (1) and (2) of class (L) of Fréchet (see [1]) if we take r-convergence as the notion of limit. But in general, it is not a topological space. We define metrizability of ranked spaces. We proved the following Proposition 1. (see [3])

Proposition 1. In two equivalment ranked spaces $(R, \mathcal{V})$ and $(R, \mathcal{U})$, $r(\pi)$-convergence and completeness are identical.

Definition 1. Consider a metric space (R, d) where we shall use (R, d) to stand for a metric space R with distance function d.

Let $\lambda_0 > \lambda_1 > \cdots\cdots > \lambda_n > \cdots\cdots \to 0$ as $n \to \infty$. If for all $p \in R$ and $n \in N$, $S(p, \lambda_n) =$

$\{q|d(p, q) \leqq \lambda_n\}$ is taken as a preneighborhood of p with rank n, then R becomes a ranked space and is called a ranked metric space. If we let $U^*(p, n)=S(p, 2^{-n})$, $\mathcal{U}_n^*=\{U^*(p, n): p\in R\}$ and $\mathcal{U}^*=\cup\{\mathcal{U}_n^*: n\in N\}$, then $(R, \mathcal{U}^*, \mathcal{U}_n^*)$ is a ranked metric space.

Deffinition 2. A ranked space $(R, \mathcal{V})$ is metrizable if we can define a distance function d in R such that the ranked metric space $(R, \mathcal{U}^*)$ obtained from the metric space (R, d) is equivalent to the ranked space $(R, \mathcal{V})$.

Proposition 2. A ranked space $(R, \mathcal{V})$ is metrizable if and only if there exists an equivalent ranked space $(R, \mathcal{U}, \mathcal{U}_n)$ with the following property.

For every point $p\in R$ and every $n\in N$, preneighborhood with rank n consists of only one preneighborhood and is denoted by $U(p, n)$. Let $\mathcal{U}_n=\{\mathcal{U}\{p, n): p\in R\}$, $\mathcal{U}=\cup\{\mathcal{U}_n: n\in N\}$ and suppose that $\{\mathcal{U}_n: n\in N\}$ satisfies the following conditions.

(1) For every $n\in N$ and every $p\in R$, we have $U(p, n)\supset U(p, n+1)$

(2) For every pair p, q of R and every $n\in N$, we have

  (i) $U(p, n)\ni q$ implies $U(q, n)\ni p$.

  (ii) $U(p, n)\cap U(q, n)\neq\phi$ implies $U(p, n-1)\ni q$.

(3) For every p of R and every sequence of preneighborhoods such that

$$U(p, 0)\supset U(p, 1)\supset \cdots\cdots \supset U(p, n)\supset \cdots\cdots, \bigcap_{n\in N} U(p, n)$$

consists of p alone. (see [3])

§ 3. Proofs by the method of ranked spaces.

In this notes we shall prove certain well known metrization theorems by the method of ranked spaces.

Some classical and still fundamental metrization theorems.

Theorem 1. (Alexandroff-Uryshone's metrization theorem)

A T_1-Space X is metrizable if and only if there is a sequence $\mathcal{U}_1, \mathcal{U}_2, \cdots\cdots$, of open coverings such that

(i) $\mathcal{U}_1>\mathcal{U}_2^*>\mathcal{U}_2>\mathcal{U}_3^*>\cdots\cdots$,

(ii) $\{S(p, \mathcal{U}_n)|n=1, 2, \cdots\cdots\}$ is a nbd base at each point p of X.

Namely $\{\mathcal{U}_i\}$ is a normal sequence. Obviously (i) may be replaced with the condition (i′), (i′) $\mathcal{U}_1>\mathcal{U}_2^{\triangle}>\mathcal{U}_2>\mathcal{U}_3^{\triangle}>\cdots\cdots$, where $\mathcal{U}^{\triangle}=\{S(p, \mathcal{U})|p\in X\}$ and $\mathcal{U}^*=\{S(U, \mathcal{U})|U\in \mathcal{U}\}$. $\mathcal{U}_2$ is a refinement of $\mathcal{U}_1$ denoted $\mathcal{U}_1>\mathcal{U}_2$ means if each set U_2 belonging $\mathcal{U}_2$ there is a set U_1 belonging $\mathcal{U}_1$ such that $U_2\subset U_1$.

Proof Evidently for every n, $\mathcal{U}_n$ is a refinement of $\mathcal{U}_{n-1}$, and we assame $\mathcal{U}_0$ consist of X alone. For every $x\in X$, put $V(x, n)=S(x, \mathcal{U}_n)$ and call it a preneighborhood of x with rank n. Put $\mathcal{V}_n=\{V(x, n)\mid x\in X\}$ and $\mathcal{V}=\cup\{\mathcal{V}_n\mid n\in N\}$. Then $(X, \mathcal{V}_n, \mathcal{V})$ becomes ranked space and $(X, \mathcal{V}_n, \mathcal{V})$ is metrizable. Because conditions (1), (3) of Prop. 2 is evidently satisfy. For condition (i) of (2) of Prop. 2, from $V(x, n)=S(x, \mathcal{U}_n)$, $V(x, n)\ni y$ implies $V(y, n)\ni x$. For condition (ii) of (2) of Prop. 2, suppose $V(x, n)\cap V(y, n)\ni z$, then there exist $U_{n-1}\in\mathcal{U}_{n-1}$ such that $U_{n-1}\supset V(z, n)\ni x, y, z$. Therefore $V(x, n-1)\supset V(z, n)$ and $V(x, n-1)\ni x, y, z$. Then $V(x, n-1)\ni y$. From (ii) of this theorem $\{S(p, \mathcal{U}_n)\}=\{V(p, n)\}$ is a nbd base in the topological space X and $(X, \mathcal{V}_n, \mathcal{V})$ is a ranked space such that r-convergence and convergence in the topological sence are identical. $\{\mathcal{V}_n\mid n\in N\}$ satisfies the conditions of Prop. 2. Therefore ranked space $\{X, \mathcal{V}\}$ is metrizable.

Theorem 2. A T_1-space X is metrizable if and only if it satisfies the following conditions. (see [4])

There exists a sequence $\{\mathcal{U}_n\mid n\in N\}$ of open coverings of X such that $\{S^2(x, \mathcal{U}_n)\mid n\in N\}$ is a nbd basis of each point p of X. Where we denote $S^2(x, \mathcal{U})=S(S(x, \mathcal{U}), \mathcal{U})$.

Proof We may assume that $\mathcal{U}_0$ consists of X alone. For $S(x, \mathcal{U}_1)$ where x is every point of X, there exists n_2 such that $S(x, \mathcal{U}_1)\supset S^2(x, \mathcal{U}_{n_2})\supset S(x, \mathcal{U}_{n_2})$. Put $V(x, 1)=S(x, \mathcal{U}_1)$ and $S(x, \mathcal{U}_{n_2})=V(x, 2)$. Then there exists n_3 such that $V(x, 1)\supset V(x, 2)\supset S^2(x, \mathcal{U}_{n_3})\supset S(x, \mathcal{U}_{n_3})$.

Put $S(x, \mathcal{U}_{n_3})=V(x, 3)\cdots\cdots S(x, \mathcal{U}_{n_i})=V(x, i)$. Put $\mathcal{V}_n=\{V(x, n)\mid x\in X\}$ and $\mathcal{V}=\cup\{\mathcal{V}_n\mid n\in N\}$. Evidently $\{X, \mathcal{V}_n, \mathcal{V}\}$ is a ranked space and it satisfies the conditions of Prop. 2.

For (1), clearly $V(x, n)\supset V(x, n+1)$; $n\in N$, for (i) of (2) $V(x, i)\ni y$ implies $V(y, i)\ni x$ for (ii) of (2) suppose $V(x, i)\cap V(y, i)\ni z$. From $S^2(x, \mathcal{U}_{n_i})=S(S(x, \mathcal{V}_{n_i}), \mathcal{U}_{n_i})$, $S^2(x, \mathcal{U}_{n_i})\ni x, y, z$ and $V(x, i-1)\supset S^2(x, \mathcal{U}_{n_i})\ni x, y, z$. We have $V(x, i-1)\ni y$.
For (3) from $S^2(x, \mathcal{U}_{n_i})\supset S(x, n_i)$ and $\{S^2(x, \mathcal{U}_n)\}$ is a nbd base, $\{S(x, \mathcal{U}_{n_i})\}$ is also a nbd base for each point x. Therefore ranked space $(X, \mathcal{V})$ is metrizable such that r-convergence and convergence in the topological sence are identical. $\{\mathcal{V}_n\mid n\in N\}$ satisfies the conditions of Prop. 2. Therefore ranked space $\{X, \mathcal{V}\}$ is metrizable.

References

1. M. Fréçht, Sur quélques points du calcul functionel, Rend. Circ. Mat. di Palermo 22 1-74 (1906)
2. F. Hausdorff, Grundzüge der Mengelehre, Leipzig 213. (1914)
3. F. Ishikawa, Metrization of ranked spaces, Canad. Math. Bull, vol. 27 (4) 385-389 (1984)
4. K. Morita, A condition for metrizability of topological space and for n-dimensionality, Sci. Rep. Tokyo-kyoiku Daigaku, Sec. A, 5 (1955) 33-36

(Received October 16, 1991)