TRANSFORMATION FORMULAS UNDER PROPER HOLOMORPHIC MAPPINGS

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ABSTRACT. A fundamental transformation rule of the inner product under proper holomorphic mappings is shown. Using this, three equivalent transformation formulas, which contain two interesting transformation theorems due to Bell[1,2, 3], and an application are given.

Let $\Gamma \equiv \Gamma_m(D, \Delta)$ be the class of proper holomorphic mappings $w=f(z)$ of a bounded domain $D \subset \mathbf{C}^n$ onto another bounded domain $\Delta \subset \mathbf{C}^n$ with order m, where Δ denotes the *basic surface* of the m sheeted Riemann region $\tilde{\Delta}$ associated with $f \in \Gamma$, so that $f(D)=\Delta$ [5]. Let us put $\omega = \{f(z) | u=0\}$ for $u(z) \equiv \det(df(z)/dz)$, then ω is a complex variety under $f \in \Gamma$ by the theorem of R. Remmert. We emphasize that the variety ω has *measure zero*, since $u \not\equiv 0$ in D. F_k and U_k $(k=1,2,\cdots,m)$ denote the m local inverses of f and their Jacobians, respectively, which are defined *locally* on $\Delta-\omega$ with $f^{-1}(\Delta-\omega) = \cup_k F_k(\Delta-\omega)$, $\Delta = f(D_k)$ and $D_k = F_k(\Delta)$, where $D = \cup_k D_k$ with $\text{meas}(D_i \cap D_j)=0$ for $i \neq j$. We note that $U_k(X) \equiv \det(dF_k(X)/dw) = 1/\det(df(x)/dz) = 1/u(x)$ and so $U_k(X)u(x)=1$ holds for $x \in D_k$ and $X \in \Delta-\omega$ with $\text{meas}(\omega)=0$, where $X=f(x)$ and $x=F_k(X)$. Furthermore, we note that $dv_X = |u(x)|^2 dv_x$ and $dv_x = |U_k(X)|^2 dv_X$. The inner product of $p, q \in L^2(D)$ (that is, each element of p and q belongs to $L^2(D)$) for a domain D is defined as usual by $\langle p, q\rangle_D \equiv \int_D p(x) q^*(x)\, dv_x\ (=\langle q, p\rangle_D^*)$. Using this, the Bergman projection P_D associated to D is denoted, as is well known, by $P_D g(z) \equiv \langle g, K_D(\ ,z)\rangle_D = \langle g, \Phi\rangle_D \Phi(z)$ for any $g \in L^2(D)$, where $\Phi \equiv {}^t(\varphi_1, \varphi_2, \cdots)$ is a complete orthonormal system in D and $K_D(z,t) \equiv \Phi^*(t)\Phi(z)$ is the Bergman kernel of D. It is trivial that $P_D s = s$ for $s \in H^2(D)$.

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Lemma. *Let $f\in\Gamma$, then we have the transformation rule*

(A) $$\langle g,h\rangle_D=\sum_{k=1}^{m}\langle U_k\cdot(g\circ F_k),\ U_k\cdot(h\circ F_k)\rangle_\Delta,$$

where $g,h\in L^2(D)$.

Proof. Noting that $U_k\cdot(p\circ F_k)\in L^2(\Delta)$ from $\|U_k\cdot(p\circ F_k)\|_\Delta^2=\|p\|_{D_k}^2\leq\|p\|_D^2<\infty$ for $p\in L^2(D)$ and $\mathrm{meas}(\omega)=0$, we have $\langle g,h\rangle_D=\sum\langle g,h\rangle_{D_k}=\sum\langle U_k\cdot(g\circ F_k),U_k\cdot(h\circ F_k)\rangle_\Delta$.

Theorem 1. *Let $f\in\Gamma$, then we have three equivalent transformation formulas:*

(1) $$u(z)(\Psi\circ f)(z)=\langle\Psi,\sum_{k=1}^{m}U_k\cdot(\Phi\circ F_k)\rangle_\Delta\Phi(z),\quad z\in D,$$

where $\Psi\equiv{}^t(\psi_1,\psi_2,\cdots)$ is a complete orthonormal system in Δ,

(2) $$u(z)K_\Delta(f(z),\tau)=\sum_{k=1}^{m}\bar{U}_k(\tau)K_D(z,F_k(\tau)),\quad z\in D \text{ and } \tau(\equiv f(t))\in\Delta-\omega \ [1,2],$$

(3) $$u\cdot((P_\Delta G)\circ f)=P_D(u\cdot(G\circ f)),\quad G\in L^2(\Delta)\ [1,2,3].$$

Proof. (1): Put $g\equiv u\cdot(\Psi\circ f)$ and $h\equiv\Phi$ in (A), then $g,\ h\in H^2(D)$. Hence, $u(z)(\Psi\circ f)(z)=\langle u\cdot(\Psi\circ f),\ \Phi\rangle_D\Phi(z)=\sum\langle U_k\cdot u\cdot(\Psi\circ f\circ F_k),\ U_k\cdot(\Phi\circ F_k)\rangle_\Delta\Phi(z)$ $=\langle\Psi,\ \sum U_k\cdot(\Phi\circ F_k)\rangle_\Delta\Phi(z)$ follows from (A).

(1)⇒(2): Put $R_k\equiv U_k\cdot(\Phi\circ F_k)$. Then R_k belongs to $H^2(\Delta)$ by the extended Riemann Removable Singularity Theorem [2], since $R_k\in L^2(\Delta)\cap H(\Delta-\omega)$ and ω is a complex variety in Δ. Multiplying both sides of (1) by $\Psi^*(\tau)(\tau\in\Delta-\omega)$ and noting that $\Psi^*(\tau)\langle\Psi,\sum R_k\rangle_\Delta\Phi(z)=[\langle\sum R_k,\Psi\rangle_\Delta\Psi(\tau)]^*\Phi(z)=\sum R_k^*(\tau)\Phi(z)$, (2) is easily obtained.

(2)⇒(3): Considering complex conjugate on both sides of (2), we have $(u\cdot((P_\Delta G)\circ f))(z)=u(z)\langle G,\Psi\rangle_\Delta\Psi(f(z))=\langle G,\bar{u}(z)K_\Delta(\ ,f(z))\rangle_\Delta=\langle G,\sum U_k\cdot K_D(F_k,z)\rangle_\Delta$ $=\sum\langle U_k\cdot(u\cdot G\circ f)\circ F_k,\ U_k\cdot(\Phi\circ F_k)\rangle_\Delta\Phi(z)=\langle u\cdot(G\circ f),\ \Phi\rangle_D\Phi(z)=P_D(u\cdot(G\circ f))(z)$ from (A) and $u\cdot G\circ f\in L^2(D)$.

(3)⇒(1): Noting that $P_\Delta\Psi=\Psi$, we have $(u\cdot(\Psi\circ f))(z)=(u\cdot((P_\Delta\Psi)\circ f))(z)$ $=P_D(u\cdot(\Psi\circ f))(z)=\langle u\cdot(\Psi\circ f),\ \Phi\rangle_D\Phi(z)=\langle\Psi,\ \sum U_k\cdot(\Phi\circ F_k)\rangle_\Delta\Phi(z)$ from (A) (see the last part in the proof of (1)).

Remark. Put $g \equiv u\cdot(G\circ f)$ in (A) for any $G\in L^2(\Delta)$, then $\langle u\cdot(G\circ f), h\rangle_D = \langle G, \sum U_k\cdot(h\circ F_k)\rangle_\Delta$ [2] is easily obtained.

Example. Let D be a unit disc and $w=f(z)=z^2\in\Gamma_2(D,D)$. Then the local inverses of f are $F_1(\tau)=\sqrt{\tau}$, $F_2(\tau)=-\sqrt{\tau}$ and their Jacobians are $U_1(\tau)=1/2\sqrt{\tau}$, $U_2(\tau)=-1/2\sqrt{\tau}$, respectively. Thus we have

$$(4)\quad \sum_{k=1}^{m}\bar{U}_k(\tau)K_D(z,F_k(\tau))=(1/2\sqrt{\bar\tau})\{1/\pi(1-\sqrt{\bar\tau}z)^2\}+(-1/2\sqrt{\bar\tau})\{1/\pi(1+\sqrt{\bar\tau}z)^2\}$$
$$=2z/\pi(1-\bar\tau w)^2=u(z)K_D(f(z),\tau)\quad \text{(see (2))}$$

for $z\in D$ and $\tau\in D-\omega$, since $u(z)=2z$ and $K_D(z,t)=1/\pi(1-\bar t z)^2$.

If we define the value in the removable singularity by the limit, (4) is valid for $\tau=0$.

Theorem 2. *Let R and Δ in $\mathbf{C}^n$ be any classical Cartan domain and a complete circular domain with center at* 0, *respectively, and let a mapping f belong to $\Gamma_m(R,\Delta)$ with $f^{-1}(0)=\{0\}$, then f must be a polynomial mapping*. (cf. [3]).

Proof. It is known that the Bergman kernel $K_R(t,z)$ of R is given by $c/\delta(t,\bar z)^s$ where s is a positive integer and δ is a polynomial of t and $\bar z$ with $\deg_t\delta=\deg_{\bar z}\delta=d$ and $\delta(t,0)=\delta(0,\bar z)\equiv 1$. (see [4] for the concrete values $s=s_i$ and $d=d_i$ with respect to each Cartan domain $R=R_i(i=\mathrm{I,II,III,IV})$).

By the reproducing property of the Bergman kernel we have

$$I(z)\equiv u(z)\langle X^\alpha, K_\Delta(X,f(z))\rangle_\Delta = u(z)f^\alpha(z),\quad X=(x_1,\cdots,x_n)\in\Delta,$$

where X^α denotes $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ for a multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)(|\alpha|\equiv\sum_{i=1}^n\alpha_i)$.
On the other hand, since Δ is a complete circular domain, then

$$X^\alpha=\sum_{\beta:|\beta|=|\alpha|}c_\beta\partial_{\bar\tau}{}^\beta K_\Delta(X,0)\equiv\lim_{\tau\to 0}\sum_\beta c_\beta\partial_{\bar\tau}{}^\beta K_\Delta(X,\tau)$$

holds (cf. [2]) for suitable constants $\{c_\beta\}$, where $\beta=(\beta_1,\cdots,\beta_n)(|\beta|\equiv\sum_{i=1}^n\beta_i)$ is a multi-index and $\partial_{\bar\tau}\equiv(\partial/\partial\bar\tau_1,\cdots,\partial/\partial\bar\tau_n)$.
Therefore we obtain

$$I(z)=\lim_{\tau\to 0}u(z)\sum_\beta c_\beta\langle\partial_{\bar\tau}{}^\beta K_\Delta(X,\tau),K_\Delta(X,f(z))\rangle_\Delta=\lim_{\tau\to 0}\sum_\beta c_\beta\partial_{\bar\tau}{}^\beta u(z)K_\Delta(f(z),\tau).$$

Using the transformation formula (2) we have

$$I(z)=\lim_{\tau\to 0}\sum_\beta c_\beta \partial_{\bar{\tau}}^{\beta}\Big(\sum_{k=1}^{m}\bar{U}_k(\tau)K_R(z,F_k(\tau))\Big)=\lim_{\tau\to 0}\sum_\beta c_\beta\Big[\partial_\tau^{\beta}\sum_{k=1}^{m}U_k(\tau)K_R(F_k(\tau),z)\Big]^{-},$$

where $\{F_k\}$ $(k=1,\cdots,m)$ are m-local inverses of f and $U_k(\tau)\equiv\det(dF_k(\tau)/d\tau)$. Now by direct calculations with Leibniz's formula, we have

$$I(z)=\lim_{\tau\to 0}\sum_\beta c_\beta\sum_{k=1}^{m}\sum_{j=0}^{\beta}[A_{kj}(\tau)\,\partial_\tau^{j}K_R(F_k(\tau),z)]^{-}\quad(\text{for } K_R(F_k(\tau),z)\equiv c\delta(F_k(\tau),\bar{z})^{-s})$$

$$=\lim_{\tau\to 0}\sum_\beta[P_\beta(\tau,\bar{z})/\textstyle\prod_k\delta(F_k(\tau),\bar{z})^{s+|\alpha|}]^{-}\equiv\sum_\beta\bar{P}_\beta(0,\bar{z})\equiv Q(z),$$

since $\lim_{\tau\to 0}\delta(F_k(\tau),\bar{z})=\delta(0,\bar{z})\equiv 1$, where $P_\beta(\tau,\bar{z})$ is a polynomial of $\bar{z}$ with holomorphic coefficients of τ and also $Q(z)$ is a polynomial of z with $\deg_z Q(z)\leq\deg_{\bar{z}}\sum_\beta P_\beta(\tau,\bar{z})\leq d((m-1)s+m|\alpha|)$. Thus each $f\in\Gamma_m(R,\Delta)$, if it exists, is a polynomial mapping (cf. [3]).

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