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Direct Application of the Unified Approach in a Semi-Infinite Medium

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Abstract

Direct application of the unified approach of invariant imbedding to a plane-parallel semi-infinite medium is considered in a simple but new manner. The integral equation satisfied by the reflection function from a semi-infinite medium is derived by virtue of the approach, For a stationary and non-stationary radiation fields, the equations obtained in the present paper coincide with those already obtained.

1. INTRODUCTION

One of the most efficient method treating the equation of radiative transfer might be the technique of invariant imbedding, initiated by Bellman and Kalaba[1]. The technique has been applied to many scientific fields and has got noteworthy results [2-4]. On extending the mathematical procedure of invariant imbedding, Wing[5] dealt with the diffuse reflection problem for a plane geometry by means of a unified approach in conjunction with the linear transport equation. The unified approach permits us to give exactly the integral equation for the reflection (scattering) and transmission functions without any ambiguity. In the theory of radiative transfer for a plane-parallel medium, the unified approach can be applied to obtain the emergent intensity from a finite medium. So far as we know, it is not applied to that of a semi-infinite medium. Since the approach derives the differential equation satisfied by the scattering and transmission functions with respect to the thickness, we can not apply it directly to the problem in a semi-infinite medium. In order to derive the equation satisfied by the reflection function from a semi-infinite medium, we need a further step that we set the derivative of the reflection function with respect to thickness to zero since it does not depends on the thickness (see Wing[2], p.51 and Bellman and Wing [3], pp. 201-202). In the present paper, we propose a method of a direct application of the unified approach to the problem in a semi-infinite medium. On contrast with the problem in a finite medium, in which a layer of infinitesimal thickness is added to the endpoint of the medium, we add the layer to the origin of the medium, where incident radiation impinges. In a quite similar manner to that used in the finite medium, we derive the integral equation satisfied by the reflection function from a semi-infinite medium. In the next section, we consider a stationary case, in which a plane-parallel homogeneous medium is illuminated by a pencil of radiation. The integral equation obtained here coincides with that of Chandrasekahr[6]. In section 3, we deals with a non-stationary radiation field in a homogeneous medium, whose surface is illuminated by a time-dependent pencil of radiation. Time-dependence of incident radiation is described by an arbitrary function. Two time parameters t_1; the mean time of a temporal capture of a phton and t_2; the mean free time, are taken into account. When the time dependence is described by the Dirac δ function or the Heaviside unit step function, the integral equation is reduced to that already obtained.

2. A SEMI INFINITE MEDIUM

Here we consider diffuse reflection of a pencil of radiation from a semi-infinite medium. This problem was discussed by Chandrasekhar[6] and solved by virtue of his H-function. With the aid of the unified approach of invariant imbedding, we treat this well known problem. Consider a semi-infinite and anisotropically scattering medium, whose surface is illuminated by a pencil of radiation of net flux πF in direction Ω_0. The specific intensity at optical depth τ in direction Ω satisfies the equation of transfer in the form

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$$\mu \frac{\partial I}{\partial \tau} = I(\tau, \Omega) - J(\tau, \Omega). \tag{2.1}$$

In the above expression, we use $\Omega = (\mu, \varphi)$, where μ is cosine of angle between the direction of radiation and the normal to the surface, φ is its azimuth and J is the source function given by

$$J(\tau, \Omega) = \frac{1}{4\pi} \int_{4\pi} p(\Omega, \Omega') I(\tau, \Omega') d\Omega'. \tag{2.2}$$

In equation (2.2) $p(\Omega, \Omega')$ is the phase function normalized to ϖ, the albedo for single scattering, as follows:

$$\frac{1}{4\pi} \int_{4\pi} p(\Omega, \Omega') d\Omega' = \varpi, \tag{2.3}$$

where we use $\int_{4\pi} = \int_{-1}^{1} d\mu \int_0^{2\pi} d\varphi$ and $\int_{2\pi} = \int_0^1 d\mu \int_0^{2\pi} d\varphi$. Denoting the intensity directed towards the surface ($\tau = 0$) by I^+ and that directed the deep interior ($\tau = \infty$) by I^-, we rewrite equation (2.1) in the following separate forms:

$$\frac{\partial}{\partial \tau} I^+(\tau, \Omega) = \frac{1}{\mu} \left(I^+(\tau, \Omega) - J^+(\tau, \Omega) \right) \equiv G^+(\tau, \Omega), \tag{2.4}$$

and

$$\frac{\partial}{\partial \tau} I^-(\tau, \Omega) = \frac{1}{\mu} \left(J^-(\tau, \Omega) - I^-(\tau, \Omega) \right) \equiv G^-(\tau, \Omega), \tag{2.5}$$

where

$$\begin{pmatrix} J^+ \\ J^- \end{pmatrix} (\tau, \Omega) = \frac{\varpi}{4\pi} \int_{2\pi} \begin{pmatrix} p^{++} & p^{+-} \\ p^{-+} & p^{--} \end{pmatrix} (\Omega, \Omega') \begin{pmatrix} I^+ \\ I^- \end{pmatrix} (\tau, \Omega') d\Omega', \tag{2.6}$$

The boundary condition for equation (2.1) is

$$I^-(0, \Omega) = \pi F \delta(\Omega - \Omega_0). \tag{2.7}$$

Following Chandrasekhar[6], we define the scattering function governing diffuse reflection through the law of diffuse reflection as follows:

$$I^+(0, \Omega; I_{\mathrm{inc}}) = \frac{1}{4\pi\mu} \int_{2\pi} S(\Omega, \Omega') I_{\mathrm{inc}}(\Omega') d\Omega', \tag{2.8}$$

where $I_{\mathrm{inc}}(\Omega)$ is incident radiation on the surface of the medium, which is given by equation(2.7). Now we start the unified approach by adding a layer of infinitesimal thickness Δ on the surface $\tau = 0$. The law of diffuse reflection (equation (2.8)) will be changed to

$$I^+(\Delta, \Omega; \hat{I}_{\mathrm{inc}}) = \frac{1}{4\pi\mu} \int_{2\pi} S(\Omega, \Omega') \hat{I}_{\mathrm{inc}}(\Omega') d\Omega', \tag{2.9}$$

because incident radiation I_{inc} is modified to $\hat{I}_{\mathrm{inc}}$ by existence of the thin layer , which is given by

$$\hat{I}_{\mathrm{inc}}(\Omega) = I^-(\Delta, \Omega). \tag{2.10}$$

From equations (2.4) and (2.5) we have

$$I^+(\Delta, \Omega; \hat{I}_{\mathrm{inc}}) = I^+(0, \Omega; I_{\mathrm{inc}}) + \Delta G^+(0, \Omega) + o(\Delta), \tag{2.11}$$

and

$$I^-(\Delta,\Omega) = I^-(0,\Omega) + \Delta G^-(0,\Omega) + o(\Delta), \tag{2.12}$$

where o means the Landau notation. Inserting equation (2.7) into equation (2.5), we have

$$G^-(0,\Omega) = \frac{1}{\mu}(J^-(0,\Omega) - \pi F\delta(\Omega-\Omega_0)). \tag{2.13}$$

The modified incident radiation given by equation (2.10) is found from equations (2.12) and (2.13) as follows:

$$\hat{I}_{\rm inc}(\Omega) = \left(1-\frac{\Delta}{\mu}\right)\pi F\delta(\Omega-\Omega_0) + \frac{\Delta}{\mu}J^-(0,\Omega) + o(\Delta). \tag{2.14}$$

In the above equation, J^- is given by equation (2.6) as

$$J^-(0,\Omega) = \frac{F}{4}\left(p^{--}(\Omega,\Omega_0) + \frac{1}{4\pi}\int_{2\pi} p^{-+}(\Omega,\Omega'')S(\Omega'',\Omega_0)d\Omega''\right). \tag{2.15}$$

Substituting equations (2.14) and (2.15) into equation (2.9), we obtain

$$I^+(\Delta,\Omega;\hat{I}_{\rm inc}) = \frac{F}{4\mu}S(\Omega,\Omega_0)(1-\frac{\Delta}{\mu_0}) + \frac{F}{4\mu}\frac{\Delta}{4\pi}\int_{2\pi} p^{--}(\Omega',\Omega_0)S(\Omega,\Omega')\frac{d\Omega'}{\mu'}$$

$$+\frac{F}{4\mu}\frac{\Delta}{16\pi^2}\int_{2\pi}\int_{2\pi} p^{-+}(\Omega',\Omega'')S(\Omega,\Omega')S(\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}. \tag{2.16}$$

Setting $\tau = 0$ in equation (2.4), we have

$$G^+(0,\Omega) = \frac{1}{\mu}(I^+(0,\Omega) - J^+(0,\Omega)). \tag{2.17}$$

From equations (2.4) and (2.8), the emergent intensity is expressed in terms of the scattering function as

$$I^+(0,\Omega) = \frac{F}{4\mu}S(\Omega,\Omega_0). \tag{2.18}$$

Substitution of equations (2.7) and (2.18) into equation (2.6) yields

$$J^+(0,\Omega) = \frac{F}{4}\left(p^{+-}(\Omega,\Omega_0) + \frac{1}{4\pi}\int_{2\pi} p^{++}(\Omega,\Omega')S(\Omega',\Omega_0)\frac{d\Omega'}{\mu'}\right). \tag{2.19}$$

Then, inserting equations (2.18) and (2.19) into equation (2.17), we have

$$G^+(0,\Omega) = \frac{F}{4\mu}\left(\frac{1}{\mu}S(\Omega,\Omega_0) - \left(p^{+-}(\Omega,\Omega_0) + \frac{1}{4\pi}\int_{2\pi} p^{++}(\Omega,\Omega'')S(\Omega'',\Omega_0)\frac{d\Omega''}{\mu''}\right)\right). \tag{2.20}$$

Since the left hand side of equation (2.11) is given by equation (2.16) and the first and the second terms on the right hand side are given by equations (2.18) and (2.20), respectively, we obtain

$$\Delta(\frac{1}{\mu}+\frac{1}{\mu_0})\frac{F}{4\mu}S(\Omega,\Omega_0) = \frac{F}{4\mu}\Delta\Big(p^{+-}(\Omega,\Omega_0) + \frac{1}{4\pi}\int_{2\pi} p^{++}(\Omega,\Omega'')S(\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}$$

$$+\frac{1}{4\pi}\int_{2\pi} p^{--}(\Omega',\Omega_0)S(\Omega,\Omega')\frac{d\Omega'}{\mu'}$$

$$+\frac{1}{16\pi^2}\int_{2\pi}\int_{2\pi} p^{-+}(\Omega',\Omega'')S(\Omega,\Omega')S(\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}\Big) + o(\Delta). \tag{2.21}$$

Dividing both sides of equation (2.21) with respect to Δ, considering the limit Δ approaches 0, we finally find the integro differential equation satisfied by the scattering function as follows

$$\begin{aligned}(\frac{1}{\mu}+\frac{1}{\mu_0})S(\Omega,\Omega_0) &= p^{+-}(\Omega,\Omega_0)+\frac{1}{4\pi}\int_{2\pi}p^{++}(\Omega,\Omega'')S(\Omega'',\Omega_0)\frac{d\Omega''}{\mu''}\\ &+\frac{1}{4\pi}\int_{2\pi}p^{--}(\Omega',\Omega_0)S(\Omega,\Omega')\frac{d\Omega'}{\mu'}\\ &+\frac{1}{16\pi^2}\int_{2\pi}\int_{2\pi}p^{-+}(\Omega',\Omega'')S(\Omega,\Omega')S(\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''},\end{aligned} \tag{2.22}$$

which coincides with that of Chandrasekhar[6].

3. THE TIME-DEPENDENT SCATTERING FUNCTION

In what follows, we consider time-dependent diffuse reflection from a semi-infinite homogeneous medium, whose surface is illuminated by a time dependent pencil of radiation of net flux $\pi F(t)$, where $F(t)$ is an arbitrary function of time. Except that all quantities in this section include time parameter t, the meanings are identical with those in the previous section. In the theory of time-dependent radiative transfer, there appear two time parameters t_1: the mean time of a photon in an absorbed state and t_2: the mean free time. The equation of radiative transfer appropriate to this case is

$$-t_2\frac{\partial}{\partial t}I+\mu\frac{\partial}{\partial\tau}I = I(t,\tau,\Omega)-J(t,\tau,\Omega), \tag{3.1}$$

where the source function is given by

$$J(t,\tau,\Omega)=\frac{1}{4\pi}\int_0^t\exp(-\frac{(t-t')}{t_1})\frac{dt'}{t_1}\int_{4\pi}p(\Omega,\Omega')I(t',\tau,\Omega')d\Omega'. \tag{3.2}$$

For the intensities directed towards the surface and that towards the deep interior, we have

$$\left(-\frac{t_2}{\mu}\frac{\partial}{\partial t}+\frac{\partial}{\partial\tau}\right)I^+(t,\tau,\Omega)=\frac{1}{\mu}\left(I^+(t,\tau,\Omega)-J^+(t,\tau,\Omega)\right)\equiv G^+(t,\tau,\Omega), \tag{3.3}$$

and

$$\left(-\frac{t_2}{\mu}+\frac{\partial}{\partial t}\frac{\partial}{\partial\tau}\right)I^-(t,\tau,\Omega)=\frac{1}{\mu}\left(J^-(t,\tau,\Omega)-I^-(t,\tau,\Omega)\right)\equiv G^-(t,\tau,\Omega), \tag{3.4}$$

where

$$\begin{pmatrix}J^+\\J^-\end{pmatrix}(t,\tau,\Omega)=\frac{1}{4\pi}\int_0^t\exp(-\frac{t-t'}{t_1})\frac{dt'}{t_1}\int_{2\pi}\begin{pmatrix}p^{++}&p^{+-}\\p^{-+}&p^{--}\end{pmatrix}(\Omega,\Omega')\begin{pmatrix}I^+\\I^-\end{pmatrix}(t',\tau,\Omega')d\Omega'. \tag{3.5}$$

Equation (3.1) should be solved subject to the initial and boundary conditions such that

$$I^{\pm}(t,\tau,\Omega)=0,\text{if}\quad t<0, \tag{3.6}$$

and

$$I^-(t,0,\Omega)=\pi F(t)\delta(\Omega-\Omega_0). \tag{3.7}$$

The emergent intensity is given by the law of diffuse reflection in the form

$$\begin{aligned}I^+(t,0,\Omega) &= I^+(t,0,\Omega;I_{\rm inc})\\ &= \frac{1}{4\pi\mu}\int_0^t\int_{2\pi}S(t-t',\Omega,\Omega')I_{\rm inc}(t',\Omega')dt'd\Omega'\end{aligned}$$

$$= \frac{1}{4\pi\mu}\int_0^t S(t-t',\Omega,\Omega_0)F(t')dt' \tag{3.8}$$

We evaluate change of the emergent intensity due to addition of a layer of infinitesimal thickness Δ to the surface. In a manner similar to that used in the previous section, the emergent intensity is modified to

$$I^+(t,\Delta,\Omega;\hat{I}_{\text{inc}}) = \frac{1}{4\pi\mu}\int_0^t\int_{2\pi} S(t-t',\Omega,\Omega')\hat{I}_{\text{inc}}(t',\Omega')dt'd\Omega', \tag{3.9}$$

where the modified incident radiation is given by

$$\hat{I}_{\text{inc}}(t,\Omega) = I^-(t,\Delta,\Omega), \tag{3.10}$$

which is written using equation (3.4) as

$$I^-(t,\Delta,\Omega) = I^-(t-\frac{t_2\Delta}{\mu},0,\Omega) + \Delta G^-(t,0,\Omega) + o(\Delta). \tag{3.11}$$

From equations (3.4) and (3.7), G^- is given by

$$G^-(t,0,\Omega) = \frac{1}{\mu}(J^-(t,0,\Omega) - \pi F(t)\delta(\Omega-\Omega_0)). \tag{3.12}$$

Now we shall evaluate the first term of the right hand side of equation (3.12) from equation (3.5) with setting $\tau = 0$

$$J^-(t,0,\Omega) = \frac{1}{4\pi}\int_0^t \exp(-\frac{t-t'}{t_1})\frac{dt'}{t_1}(\int_{2\pi} p^{-+}(\Omega,\Omega')I^+(t',0,\Omega')d\Omega'$$
$$+\int_{2\pi} p^{--}(\Omega,\Omega')I^-(t',0,\Omega')d\Omega'). \tag{3.13}$$

Substituting equations (3.7) and (3.8) into equation (3.13) we have

$$J^-(t,0,\Omega) = \frac{1}{4\pi}\int_0^t \exp(-\frac{t-t'}{t_1})\frac{dt'}{t_1}\int_{2\pi} p^{-+}(\Omega,\Omega'')\frac{d\Omega''}{4\mu''}$$
$$\times\int_0^{t'} S(t'-t'',\Omega'',\Omega_0)F(t'')dt''$$
$$+\frac{1}{4\pi}\int_0^t \exp(-\frac{t-t'}{t_1})\pi F(t')\delta(\Omega'-\Omega_0)p^{--}(\Omega,\Omega')d\Omega'$$
$$= \frac{1}{4\pi}\int_0^t \exp(-\frac{t-t'}{t_1})\frac{dt'}{t_1}\int_0^{t'} F(t'')dt''$$
$$\times\frac{1}{4\pi}\int_{2\pi} p^{-+}(\Omega,\Omega'')S(t'-t'',\Omega'',\Omega_0)\frac{d\Omega''}{\mu''}$$
$$+\frac{1}{4}p^{--}(\Omega,\Omega_0)\int_0^t \exp(-\frac{t-t'}{t_1})F(t')\frac{dt'}{t_1}. \tag{3.14}$$

Inserting equations (3.7), (3.11), (3.12) and (3.13) into equation (3.14), we obtain the modified incident radiation in the form

$$\hat{I}_{\mathrm{inc}}(t,\Omega)=\pi F(t-\frac{t_2\Delta}{\mu})\delta(\Omega-\Omega_0)+\frac{\Delta}{4\mu}\int_0^t\exp(-\frac{t-t'}{t_1})\frac{dt'}{t_1}\int_0^{t'}F(t'')dt''$$
$$\times\frac{1}{4\pi}\int_{2\pi}p^{-+}(\Omega,\Omega'')S(t'-t'',\Omega'',\Omega_0)\frac{d\Omega''}{\mu''}$$
$$+\frac{\Delta}{4\mu}p^{--}(\Omega,\Omega_0)\int_0^t\exp(-\frac{t-t'}{t_1})F(t')\frac{dt'}{t_1}-\frac{\Delta}{\mu}F(t)\delta(\Omega-\Omega_0)+o(\Delta). \tag{3.15}$$

Using equation (3.15) in equation (3.9), we find the modified emergent intensity as follows:

$$I^+(t,\Delta,\Omega;\hat{I}_{\mathrm{inc}})=\frac{1}{4\mu}\int_0^t S(t-t',\Omega,\Omega_0)F(t'-\frac{t_2\Delta}{\mu_0})dt'$$
$$+\frac{\Delta}{4\mu}\int_0^t dt'\int_0^{t'}\exp(-\frac{t'-t''}{t_1})\frac{dt''}{t_1}\int_0^{t''}F(t''')dt'''$$
$$\times\frac{1}{16\pi^2}\int_{2\pi}\int_{2\pi}p^{-+}(\Omega',\Omega'')S(t-t',\Omega,\Omega')S(t''-t''',\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}$$
$$+\frac{\Delta}{4\mu}\int_0^t dt'\int_0^{t'}\exp(-\frac{t'-t''}{t_1})F(t'')\frac{dt''}{t_1}\frac{1}{4\pi}\int_{2\pi}p^{--}(\Omega,\Omega_0)S(t-t',\Omega,\Omega')\frac{d\Omega'}{\mu'}$$
$$+\frac{\Delta}{4\mu\mu'}\int_0^t S(t-t',\Omega,\Omega_0)F(t')dt'. \tag{3.16}$$

On the other hand, from equation (3.3), it follows that

$$I^+(t-\frac{t_2}{\mu}\Delta,\Delta,\Omega)=I^+(t,0,\Omega)+\frac{\Delta}{\mu}[I^+(t,0,\Omega)-J^+(t,0,\Omega)]+o(\Delta), \tag{3.17}$$

where $I^+(t,0,\Omega)$ is given by equation (3.8), and $J^+(t,0,\Omega)$ is evaluated by equation (3.5) with equations (3.8) and (3.9) in the following form:

$$J^+(t,0,\Omega)=\frac{1}{4}\int_0^t\exp(-\frac{t-t'}{t_1})\frac{dt'}{t_1}\frac{1}{4\pi}\int_{2\pi}p^{++}(\Omega,\Omega')\frac{d\Omega'}{\mu'}\int_0^{t'}S(t'-t'',\Omega',\Omega_0)F(t'')dt''$$
$$+\frac{1}{4}p^{+-}(\Omega,\Omega_0)\int_0^t\exp(-\frac{t-t'}{t_1})F(t')\frac{dt'}{t_1}. \tag{3.18}$$

Then equation (3.17) becomes

$$I^+(t-\frac{t_2}{\mu}\Delta,\Delta)=\frac{1}{4\mu}\int_0^t S(t-t',\Omega,\Omega_0)F(t')dt'+\frac{\Delta}{4\mu^2}\int_0^t S(t-t',\Omega,\Omega_0)F(t')dt'$$
$$-\frac{\Delta}{4\mu}\int_0^t\exp(-\frac{t-t'}{t_1})\frac{dt'}{t_1}\frac{1}{4\pi}\int_{2\pi}p^{++}(\Omega,\Omega')\frac{d\Omega'}{\mu'}\int_0^{t'}S(t'-t'',\Omega',\Omega_0)F(t'')dt''$$
$$-\frac{\Delta}{4\mu}p^{+-}(\Omega,\Omega_0)\int_0^t\exp(-\frac{t-t'}{t_1})F(t')\frac{dt'}{t_1}+o(\Delta). \tag{3.19}$$

Subtraction equation (3.16) from equation (3.19) yields

$$I^+(t,\Delta,\Omega)-I^+(t-\frac{t_2}{\mu}\Delta,\Delta,\Omega)=-\frac{\Delta}{4\mu}\int_0^t S(t-t',\Omega,\Omega_0)\{F(t')-F(t'-\frac{t_2\Delta}{\mu_0})\}dt'$$

$$-\frac{\Delta}{4\mu}(\frac{1}{\mu}+\frac{1}{\mu_0})\int_0^t S(t-t',\Omega,\Omega_0)F(t')dt' + \frac{\Delta}{4\mu}p^{+-}(\Omega,\Omega_0)\int_0^t \exp(-\frac{t-t'}{t_1})F(t')\frac{dt'}{t_1}$$
$$+\frac{\Delta}{4\mu}\int_0^t dt'\int_0^{t'}\exp(-\frac{t'-t''}{t_1})F(t'')\frac{dt''}{t_1}\frac{1}{4\pi}\int_{2\pi}p^{--}(\Omega,\Omega_0)S(t-t',\Omega,\Omega')\frac{d\Omega'}{\mu'}$$
$$+\frac{\Delta}{4\mu}\int_0^t \exp(-\frac{t-t'}{t_1})\frac{dt'}{t_1}\frac{1}{4\pi}\int_{2\pi}p^{++}(\Omega,\Omega')\frac{d\Omega'}{\mu'}\int_0^{t'}S(t'-t'',\Omega',\Omega_0)F(t'')dt''$$
$$+\frac{\Delta}{4\mu}\int_0^t dt'\int_0^{t'}\exp(-\frac{t'-t''}{t_1})\frac{dt''}{t_1}\int_0^{t''}F(t''')dt'''$$
$$\times\frac{1}{16\pi^2}\int_{2\pi}\int_{2\pi}p^{-+}(\Omega',\Omega'')S(t-t',\Omega,\Omega'')S(t''-t''',\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}+o(\Delta). \tag{3.20}$$

The left hand side of the above equation reads

$$\frac{t_2\Delta}{\mu}\frac{\partial}{\partial t}I^+(t,\Delta,\Omega) = \frac{t_2\Delta}{\mu}\frac{1}{4\mu}\int_0^1 \frac{\partial}{\partial t}S(t-t',\Omega,\Omega_0)F(t')dt', \tag{3.21}$$

and the first term of the right hand side reads

$$\frac{1}{4\mu}\int_0^t S(t-t',\Omega,\Omega_0)\frac{t_2\Delta}{\mu_0}F'(t)dt = \frac{\Delta t_2}{4\mu\mu_0}\int_0^t \frac{\partial}{\partial t}S(t-t',\Omega,\Omega_0)F(t')dt'. \tag{3.22}$$

Considering the limit that Δ approaches 0, where $o(\Delta)/\Delta \to 0$, we have

$$t_2(\frac{1}{\mu}+\frac{1}{\mu_0})\int_0^t \frac{\partial}{\partial t}S(t-t',\Omega,\Omega_0)F(t')dt' + (\frac{1}{\mu}+\frac{1}{\mu_0})\int_0^t S(t-t',\Omega,\Omega_0)F(t')dt'$$
$$= p^{+-}(\Omega,\Omega_0)\int_0^t \exp(-\frac{t'-t''}{t_1})F(t'')\frac{dt''}{t_1}\frac{1}{4\pi}\int_{2\pi}p^{--}(\Omega,\Omega_0)S(t-t',\Omega,\Omega')\frac{d\Omega'}{\mu'}$$
$$+\int_0^t \exp(-\frac{t-t'}{t_1})\frac{dt'}{t_1}\int_0^{t'}F(t'')dt''\frac{1}{4\pi}\int_{2\pi}p^{++}(\Omega,\Omega'')S(t'-t'',\Omega'',\Omega_0)\frac{d\Omega''}{\mu''}$$
$$+\int_0^t dt'\int_0^{t'}\exp(-\frac{t'-t''}{t_1})\frac{dt''}{t_1}\int_0^{t''}F(t''')dt'''$$
$$\times\frac{1}{16\pi^2}\int_{2\pi}\int_{2\pi}p^{-+}(\Omega',\Omega'')S(t-t',\Omega,\Omega')S(t''-t''',\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{\Omega''}{\mu''} \tag{3.23}$$

The convolution with $F(t)$ can be canceled out from both sides of equation (3.23), and hence we have the desired differential equation satisfied by the scattering function as follows:

$$(1+t_2\frac{\partial}{\partial t})(\frac{1}{\mu}+\frac{1}{\mu_0})S(t,\Omega,\Omega_0) = p^{+-}(\Omega,\Omega_0)\frac{e^{-t/t_1}}{t_1} * \left[\delta(t)p^{+-}(\Omega,\Omega_0)\right.$$
$$+\frac{1}{4\pi}\int_{2\pi}p^{--}(\Omega,\Omega')S(t,\Omega,\Omega')\frac{d\Omega'}{\mu'}+\frac{1}{4\pi}\int_{2\pi}p^{++}(\Omega,\Omega'')S(t,\Omega'',\Omega_0)\frac{d\Omega''}{\mu''}$$
$$\left.+\frac{1}{16\pi^2}\int_{2\pi}\int_{2\pi}p^{-+}(\Omega',\Omega'')S(t,\Omega,\Omega') * S(t,\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{\Omega''}{\mu''}\right], \tag{3.24}$$

where $*$ means the convolution.

Thus we obtain an integro-differential equation for the scattering function, which is defined by equation (3.8). This equation is the same that derived by Matsumoto[7], where time-dependence of incident radiation is given by the Dirac δ function. Since the exact solution is known [7], we can find from equation (3.8) the scattering function for arbitrary $F(t)$. As an example we shall consider the case $F(t) = u(t)$, where $u(t)$ is the Heaviside's unit step function. Setting $F(t) = u(t)$ in equation (3.8), we have

$$I^{+}(t,0,\Omega) = \frac{1}{4\mu}\int_0^t S(t,\Omega,\Omega_0). \tag{3.25}$$

Setting

$$R(t,\Omega,\Omega_0) = \int_0^t S(t,\Omega,\Omega_0)dt, \tag{3.26}$$

from equation (3.24) we obtain the integro-differential equation for the R-function as follows:

$$(1+\frac{\partial}{\partial t})R(t,\Omega,\Omega_0) = \frac{e^{-t/t_1}}{t_1} * \big[p^{+-}(\Omega,\Omega_0)u(t)$$

$$+\frac{1}{4\pi}\int_{2\pi} p^{--}(\Omega,\Omega_0)R(t,\Omega,\Omega') + \frac{1}{4\pi}\int_{2\pi} p^{++}(\Omega,\Omega'')R(t,\Omega'',\Omega_0)\frac{d\Omega''}{\mu'}$$

$$+\frac{1}{16\pi^2}\frac{\partial}{\partial t}\int_{2\pi}\int_{2\pi} p^{-+}(\Omega',\Omega'')R(t,\Omega,\Omega') * R(t,\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}\big]. \tag{3.27}$$

This equation coincides with that obtained by Matsumoto[8].

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