On Conakayama Rings

Yoshito YUKIMOTO

1. Introduction

A module M is called uniserial if the submodule lattice of M is linearly ordered. A ring R is called right (resp. left) nakayama if every projective indecomposable right (resp. left) R−module is uniserial. A ring R is called right (resp. left) conakayama if every injective indecomposable right (resp. left) R−module is uniserial.

If a ring is right nakayama (resp. right conakayama) and left nakayama (resp. left conakayama), simply we call it nakayama (resp. conakayama).

In this note we show that an artinian ring is conakayama if and only if it is nakayama.

We denote the Jacobson radical of a ring by J in case that the ring is clear in the context, and by M^* we denote the dual module Hom (M,R) of a module M.

For an artinian ring R, the following three propositions are well-known:

(1) If R/J^2 is nakayama, then R is nakayama.

(2) If R is nakayama, then R is conakayama.

(3) If R is conakayama, then R/J^2 is conakayama.

Therefore it is sufficient for us to show

(4) If R/J^2 is conakayama, then R/J^2 is nakayama.

2. Simple modules over a ring with $J^2=0$

Suppose that R is an artinian ring with $J^2=0$. Then Soc (R) is the direct sum of J and simple projective modules. If S is a non−projective simple module, then the image of any element f of S^* is included in J. Therefore S^* is simple or zero.

3. Dual modules of a projective module of length ≧ 2

Suppose that R is an artinian ring with the Jacobson radical J, and that $J^2=0$.

Let $0 \rightarrow K \rightarrow P \xrightarrow{f} S \rightarrow 0$ be an exact sequence of non-projective simple module S, its projective cover P, and the kernel K of the projective cover. The following exact sequence is derived from this exact sequence:

$$0 \to S^* \xrightarrow{f^*} P^* \to K^*$$

Since K is a (finite) direct sum of simple projective modules and simple non−projective module, the dual module K^* is a direct sum of projective modules and simple modules (cf. section 2) . If Im (f^*) is neither simple nor zero, then Im (f^*) is isomorphic to the dual to a projective direct summand of K^*,. and P is a direct summand of K, which contradicts to the lengths of P and K. Therefore Im (f^*) is simple or zero. On the other hand, the dual module S^* of S is simple or zero. Hence the local module P^* is uniserial. Thus we have the following lemma.

Lemma 1. If a ring R is an artinian ring with $J^2=0$, then the dual R−module of a projective module of length $\geqq 2$ is uniserial.

4. Extensions

Assume that R is an artinian (right) conakayama ring with $J^2=0$. Let S, P be projective indecomposable right R−modules, $0\to S \to P \to C \to 0$ be an exact sequence.

Now consider Ext (C,R) .

If $0\to Q \to X \to C \to 0$ is a non−split exact sequence with a projective indecomposable module Q. There are only two cases. One is the case that Q is not superflous in X. The other is the case that Q is superflous in X. In the former case $X=Q+L$ with a submodule $L \ngeq Q$ of X, where some factor module of L is isomorphic to C. Let M be a simple submodule in $Q \cap L \neq O$ and N be a complement of M in the socle of X. Then the factor module of X by N is a colocal module not local, which contradicts to our assumption. Hence Q is superflous in X i.e. $Q \leqq XJ$. Since $J^2=0$, the projective indecomposable module Q is simple. In other words, if the length of a projective indecomposable module Q is $\geqq 2$, then Ext $(Q,C)=0$.

Let the following diagram be commutative:

$$\begin{array}{ccccccccc}
0 & \to & Q & \to & X & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & P & \to & Y & \to & C & \to & 0,
\end{array}$$

where both rows are exact and the homomorphism from the projective indecomposable module Q to another projective indecomposable module P is represented by the left multiplication of an element in J. If the homomorphism is not zero then the length of P is $\geqq 2$. It follows that $J \cdot$ Ext $(C,R)=0$.

Lemma 2. The dual module S^* to a simple projective module S is uniserial.

Proof. If S is not subisomorphic to any projective indecomposable module of another type, then S^* is simple.

Let $0 \to S \to P \to C \to 0$ be an exact sequence with simple projective module S, projective indecomposable module P and the cokernel $C \neq 0$. From this exact sequence the following exact sequence is derived:

$$0 \to C^* \xrightarrow{a} P^* \xrightarrow{b} S^* \to \mathrm{Ext}\,(C,R) \to 0.$$

If Ext (C,R) =0, then P^* is isomorphic to S^* and P is isomorphic to S, which contradicts to $C \neq 0$. Since S^* is local and Ext (C,R) is semisimple in the above sequence, the derived module Ext (C,R) is simple and the kernel Ker (b) of b is semisimple. Since P^* is local and Ker (b) (=Im (a)) is semisimple, the submodule Ker (b) of S^* is simple or zero. Therefore the local module S^* is uniserial.

5. Main theorem

By Lemma 1 and Lemma 2 we have the following proposition and theorem.

Proposition 3. If R is an artinian conakayama ring with $J^2 = 0$, then R is nakayama.

Proof. By Lemma 1 and Lemma 2, every projective left R−module is uniserial, and every projective right module is uniserial by symmetry.

Theorem 4. An artinian ring is conakayama if and only if it is nakayama.

Proof. It follows from Proposition 3 and the three propositions (1),(2),(3) in the introduction.

REFERENCES

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