

# On a Diagonalization of Matrices over Regular Ring II

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ABSTRACT. An element  $x$  in a ring  $R$  is called right (resp. left) invertible if there exists  $y \in R$  such that  $xy = 1$  (resp.  $yx = 1$ ). An element in a ring  $R$  is called invertible if it is right invertible and left invertible. In the previous paper [2] we show that, for any right invertible matrix  $X$  in the ring  $M_2(R)$  of all (2,2)-matrices over a regular ring  $R$ , there exist an element  $Y \in M_2(R)$  and an invertible matrix  $V \in M_2(R)$  such that  $XVY = I$  (identity matrix) and  $YXV$  is a diagonal matrix. In this paper we generalize the result of (2,2)-matrices to that of (n,n)-matrices.

## 1. Introduction

An element  $a$  in a ring  $R$  is said to be a generalized inverse of  $x \in R$  if  $x = xax$  and  $a = axa$ . A ring  $R$  is called (von Neumann) regular if for any element  $x \in R$  there exists a generalized inverse of  $x$ . This generalized inverse is not uniquely determined in general. The set of all generalized inverses of an element  $x$  in a regular ring is denoted by  $x^*$ . We show that, for any right invertible (n,n)-matrix  $M$  over a regular ring  $R$ , there exist an (n,n)-matrix  $N$  and an invertible (n,n)-matrix  $Q$  such that  $MQN = I$  and  $NMQ$  is a diagonal matrix.

## 2. Semidiagonal matrices

We denote the unit matrix by  $I$ . By  $I_{ij}$  we denote the matrix whose (i,j)-entry is 1 and the other entries are 0. The matrix  $I + xI_{ij}$  is denoted by  $T_{ij}(x)$  for  $i \neq j$ . A matrix  $A$  is said to be right transformed to a matrix  $B$  if there exist  $T_{i_1j_1}(x_1), T_{i_2j_2}(x_2), \dots, T_{i_tj_t}(x_t)$  such that

$$AT_{i_1j_1}(x_1)T_{i_2j_2}(x_2) \dots T_{i_tj_t}(x_t) = B.$$

A matrix  $A$  is said to be bilaterally transformed to a matrix  $B$  if there exist  $T_{i_1j_1}(x_1), \dots, T_{i_sj_s}(x_s), T_{i_{s+1}j_{s+1}}(x_{s+1}), \dots, T_{i_{s+t}j_{s+t}}(x_{s+t})$  such that

$$T_{i_1j_1}(x_1) \dots T_{i_sj_s}(x_s)AT_{i_{s+1}j_{s+1}}(x_{s+1}) \dots T_{i_{s+t}j_{s+t}}(x_{s+t}) = B.$$

An (m,n)-matrix  $[x_{ij}]$  over a regular ring is called semidiagonal if there exist  $a_{ij} \in x_{ij}^*$  such that

$$(i) \quad a_{ki}x_{kj} = 0 \quad (i \neq j, 1 \leq i, j \leq n, 1 \leq k \leq m)$$

and

$$(ii) \quad x_{ik}a_{jk} = 0 \quad (i \neq j, 1 \leq i, j \leq m, 1 \leq k \leq n).$$

Especially a row vector  $[x_1 \dots x_n]$  is called semidiagonal if there exist  $a_i \in x_i^*$  such that  $a_ix_j = 0 \quad (i \neq j)$ .

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**Lemma 1.** Every vector  $[x_1 \dots x_n]$  over a regular ring is right transformed to a semidiagonal vector.

*Proof.* We prove by induction on  $n$ .

For  $n = 1$  there is nothing to be shown. By the hypothesis of induction there exist  $a_i \in x_i^*$  such that  $a_i x_j = 0$  ( $i \neq j$ ) for  $i, j < n$ . Multiplying  $[x_1 \dots x_n]$  by  $T_{1n}(-a_1 x_n) T_{2n}(-a_2 x_n) \dots T_{n-1,n}(-a_{n-1} x_n)$  on the right, we get

$$[x_1 \ x_2 \ \dots \ x_{n-1} \ (1 - x_1 a_1 - \dots - x_{n-1} a_{n-1}) x_n].$$

Let  $y_n = (1 - x_1 a_1 - \dots - x_{n-1} a_{n-1}) x_n$ ,  $a_n$  a generalized inverse of  $y_n$ , and  $b_n = a_n (1 - x_1 a_1 - \dots - x_{n-1} a_{n-1})$ . Then  $b_n \in y_n^*$ ,  $a_i y_n = 0$  ( $i < n$ ) and  $b_n x_j = 0$  ( $j < n$ ). Therefore  $[x_1 \dots x_{n-1} \ y_n]$  has the required property.

**Lemma 2.** If  $[x_{ij}]$  is an  $(m, n)$ -matrix with  $a_{ij} \in x_{ij}^*$  satisfying  $a_{ki} x_{kj} = 0$  ( $i \neq j$ ,  $1 \leq i, j \leq n$ ,  $1 \leq k \leq m-1$ ) and  $x_{ik} a_{jk} = 0$  ( $i \neq j$ ,  $1 \leq i, j \leq m$ ,  $1 \leq k \leq n$ ). Then  $[x_{ij}]$  is right transformed to a semidiagonal matrix.

*Proof.* Since the multiplication by  $T_{1n}(-a_1 x_n) T_{2n}(-a_2 x_n) \dots T_{n-1,n}(-a_{n-1} x_n)$  on the right does not vary the upper  $m-1$  rows, the proof is the same of Lemma 1.

**Theorem 3.** Every  $(m, n)$ -matrix  $[x_{ij}]$  is bilaterally transformed to a semidiagonal matrix.

*Proof.* We prove by induction of  $m$ .

Lemma 1 is the case of  $m = 1$ .

By the hypothesis of induction there exist  $a_{ij} \in x_{ij}^*$  ( $1 \leq i \leq m-1$ ,  $1 \leq j \leq n$ ) such that  $a_{ki} x_{kj} = 0$  ( $i \neq j$ ,  $1 \leq i, j \leq n$ ,  $1 \leq k \leq m-1$ ) and  $x_{ik} a_{jk} = 0$  ( $i \neq j$ ,  $1 \leq i, j \leq m-1$ ,  $1 \leq k \leq n$ ). Multiply  $[x_{ij}]$  by

$$T_{m1}(-x_{mj} a_{1j}) T_{m2}(-x_{mj} a_{2j}) \dots T_{m,m-1}(-x_{mj} a_{m-1,j})$$

on the left for  $j = 1, \dots, n$ . Then the  $(m, j)$ -entry is changed to  $y_{mj} = x_{mj} (1 - a_{1j} x_{1j} - \dots - a_{m-1,j} x_{m-1,j})$  without any side effect on the other entries. Let  $a_{mj}$  be a generalized inverse of  $y_{mj}$ , and  $b_{mj} = (1 - a_{1j} x_{1j} - \dots - a_{m-1,j} x_{m-1,j}) a_{mj}$ . Then  $b_{mj} \in y_{mj}^*$  and the matrix

$$\begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m-1,1} & \dots & x_{m-1,n} \\ y_{m1} & \dots & y_{mn} \end{bmatrix}$$

is applicable to Lemma 2 (choose  $b_{mj}$  as generalized inverse of  $y_{mj}$ ), which completes the proof.

### 3. Diagonalization

Let  $X = [x_{ij}]$  be a semidiagonal  $(m, n)$ -matrix over a regular ring  $R$ ,  $a_{ij} \in x_{ij}^*$  generalized inverses satisfying the condition (i) and (ii) in the definition of semidiagonal matrix, and  $A = [a_{ij}]$ . Then  $X^t A$  is a diagonal matrix with diagonal elements

$$x_{i1} a_{i1} + \dots + x_{in} a_{in} \quad (i = 1, \dots, m),$$

which are idempotent elements. The matrix  ${}^tA X$  is also a diagonal matrix with diagonal elements

$$a_{1j}x_{1j} + \cdots + a_{mj}x_{mj} \quad (j = 1, \dots, n),$$

which are idempotent elements.

If  $X$  is  $(n,n)$ -matrix and right invertible, then

$$x_{i1}r_1 + \cdots + x_{in}r_n = 1$$

for some  $r_1, \dots, r_n \in R$ . The multiplication of the both sides of the equation by  $x_{ij}a_{ij}$  makes  $x_{ij}r_i = x_{ij}a_{ij}$  and we have  $x_{i1}a_{i1} + \cdots + x_{in}a_{in} = 1$ . Therefore any right invertible matrix is bilaterally transformed to a right invertible semidiagonal matrix. Hence, for any right invertible  $(n,n)$ -matrix  $M$ , there exist invertible matrices  $P, Q$ , right invertible matrix  $X$  and left invertible matrix  $Y$  such that  $PMQ = X$ ,  $XY = I$  and  $YX$  is diagonal. Then  $MQYP = I$  and  $YPMQ$  is diagonal. We have  $MQN = I$  by putting  $N = YP$ , and  $NMQ$  is diagonal.

#### REFERENCES

1. K. R. Goodearl, *Von Neumann Regular Rings*, Pitman, London, 1979.
2. Y. Yukimoto, *On a Diagonalization of Matrices over Regular Ring*, Memoir of Fukui Univ. Tech. **24** (1993), 251–253.

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