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Mathematical Verification of Finite Order Principles of Invariance I. Isotropic Scattering in a Homogeneous Atmosphere

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Abstract

With the aid of the equation of radiative transfer satisfied by a specific intensity of finite order of scattering, in a finite homogeneous atmosphere which scatters radiation isotropically, two functional relations satisfied by the finite order source function are derived exactly. The complete set of the finite order principles of invariance is proved in virtue of the functional relations.

1. Introduction

One of the most efficient method solving the equation of radiative transfer is the successive scattering (or order of scattering) theory initiated by van de Hulst(1948). It deals with equations which describe photons scattered one, two or n times in the medium. This approach is employed, in one hand, to obtain analytical expressions of specific intensities for relatively small n (cf. Irvine 1969, Hovenir 1969, 1971, Kawabata and Ueno 1989, Matsumoto 1996). On the other hand, in virtue of this approach, we obtain a set of recurrent equations satisfied n-th order specific intensities, which are really adequate to numerical calculation, instead to solving the non linear integro differential equations. Tanaka(1966) obtained numerical values of reflected and transmitted intensities from a finite atmosphere. When the optical thickness tends to infinity, a recurrent relation is obtained by Uesugi and Irvine(1969) who have an asymptotic expression for the high orders of scattering, which is developed by van de Hulst(1970).

In the theory of time dependent radiative transfer, the order of scattering theory demonstrate it's real ability. So far as we know, this is the only method obtaining the exact solution to the time dependent equation of transfer in a semi-infinite homogeneous atmosphere(cf. Matsumoto 1976, Ganapol and Matsumoto 1986, Matsumoto 1994).

In the first paper of a series of papers within the medical context of radiation dosimetry, Bellman, Ueno and Vasudevan (1972) stated the principles of invariance, which govern invariance of laws of diffuse reflection and transmission under an addition or subtraction of an atmosphere of finite thickness, for finite order scattering and transmission functions. Successively, Fymat and Ueno (1974) formulated the complete set of principles of invariance in a finite inhomogeneous atmosphere, in taking account of polarization. In both papers, the principles are stated physically, without any mathematical proof.

The purpose of this paper is to prove the principles of invariance for finite order reflection and transmission functions which describe the finite order emergent intensities

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from a homogeneous and isotropically scattering atmosphere of finite optical thickness. Starting with the equation of transfer satisfied by a finite order specific intensity, we obtain two functional relations satisfied by the finite order source function. The complete set of the finite order principles of invariance follows from the functional relations.

2. Basic Equations

Consider a finite, plane-parallel, non-emitting and homogeneous atmosphere which scatters radiation isotropically. Let a beam of radiation of net flux πF per unit area normal to itself in direction μ_0 be incident on the upper surface ($\tau = 0$). We denote the n-times scattered specific intensity, at optical depth τ, in direction μ directed towards the lower surface ($\tau = \tau_1$) by $I(n, \tau, +\mu)$ and that directed towards the upper surface ($\tau = 0$) by $I(n, \tau, -\mu)$, respectively, where μ_0 or μ is the cosine of the angle between the direction of radiation and the outward normal to the surface of the atmosphere. The equation of radiative transfer satisfied by the n-th order (that is to say, n times scattered) specific intensity is written in the form (see Bellman, Ueno and Vasudevan;1972)

$$\mu\frac{\partial}{\partial\tau}I(n, \tau \pm \mu) \pm I(n, \tau, \pm\mu) = J(n, \tau, \mu_0), \tag{2.1}$$

with boundary conditions,

$$I(n, \tau_1, +\mu) = 0, \tag{2.1}$$

and

$$I(n, 0, -\mu) = 0, \tag{2.2}$$

where $J(n, \tau, \mu_0)$ is the n-th order source function given by

$$J(n, \tau, \mu_0) = \frac{\varpi}{2}\int_0^1 I(n-1, \tau, +\mu)d\mu + \frac{\varpi F}{4}e^{-\tau/\mu_0}\delta(n, 1). \tag{2.3}$$

In the above expression, ϖ is the albedo for single scattering ($0 \leq \varpi \leq 1$) and δ is Kronecker's δ function. The total intensity and the total source function are given by

$$I(\tau, \pm\mu) = \sum_{n=1}^{\infty} \varpi^n I(n, \tau, \pm\mu), \tag{2.4}$$

and

$$J(\tau, \mu_0) = \sum_{n=1}^{\infty} \varpi^n J(n, \tau, \mu_0). \tag{2.5}$$

Solving equation (2.1) with respect to $I(n, \tau, \pm\mu)$, we have

$$I(n, \tau, +\mu) = \int_0^{\tau} J(n, t, \mu_0)e^{-(\tau-t)/\mu}\frac{dt}{\mu}, \tag{2.6}$$

and

$$I(n, \tau, -\mu) = \int_{\tau}^{\tau_1} J(n, t, \mu_0)e^{-(t-\tau)/\mu}\frac{dt}{\mu}. \tag{2.7}$$

Substituting equations (2.5) and (2.6) into (2.3), we obtain a recurrence relation for the n-th order source function as follows:

$$J(n,\tau,\mu_0)=\frac{\varpi}{2}\int_0^{\tau_1}J(n-1,t,\mu_0)E_1(|\tau-t|)dt+\frac{\varpi F}{4}e^{-\tau/\mu_0}\delta(n,1), \tag{2.8}$$

where E_1 is the first order exponential integral function such that

$$E_1(x)=\int_0^1 e^{-x/\mu}\frac{d\mu}{\mu}. \tag{2.9}$$

The first order source function is

$$J(1,\tau,\mu_0)=\frac{\varpi F}{4}e^{-\tau/\mu_0}. \tag{2.10}$$

From equation (2.8), we can see that the n-th order source function depends not only on τ and μ_0 but also on τ_1. Then, hereafter, we write it as $J(n,\tau,\tau_1,\mu_0)$.

The reflection and transmission functions of order n are defined as

$$I(n,\tau_1,-\mu)=\frac{F}{4\mu}S(n,\tau_1,\mu,\mu_0), \tag{2.11}$$

and

$$I(n,0,+\mu)=\frac{F}{4\mu}T(n,\tau_1,\mu,\mu_0). \tag{2.12}$$

From equations (2.6) and (2.7) we have

$$S(n,\tau_1,\mu,\mu_0)=\frac{4}{F}\int_0^{\tau_1}J(n,t,\tau_1,\mu_0)e^{-t/\mu}dt, \tag{2.13}$$

and

$$T(n,\tau_1,\mu,\mu_0)=\frac{4}{F}\int_0^{\tau_1}J(n,t,\tau_1,\mu_0)e^{-(\tau_1-\tau)/\mu}dt. \tag{2.14}$$

3. The Functional Relations for the Finite Order Source Function

For the n-th order source function we have the functional relations stated in the following theorem.

Theorem. *For the n-th order source function, the following two functional relations hold:*

$$J(n,t+\tau,\tau_1,\mu_0)=e^{-\tau/\mu_0}J(n,t,\tau_1-\tau,\mu_0)$$
$$+\frac{2}{F}\sum_{k=1}^{n-1}\int_0^1 J(k,t,\tau_1-\tau,\mu')I(n-k,\tau,\mu')d\mu', \quad \text{if} \quad 0\le t\le\tau_1-\tau, \tag{3.1}$$

and

$$J(n,t,\tau_1,\mu_0)=J(n,t,\tau,\mu_0)$$
$$+\frac{2}{F}\sum_{k=1}^{n-1}\int_0^1 J(k,\tau-t,\tau,\mu')I(n-k,\tau,-\mu')d\mu'. \quad \text{if} \quad 0\le t\le\tau\le\tau_1. \tag{3.2}$$

Proof. We shall prove by induction with respect to n. First we consider equation (3.1). For $n=1$, equation (3.1) reduces to

$$J(1,t+\tau,\tau_1,\mu_0)=e^{-\tau/\mu_0}J(1,\tau_1-\tau,\mu_0).$$

This is evident from equation (2.10). Suppose that equation (3.1) is valid up to n. From equation (2.8) we have

$$J(n+1,t+\tau,\tau_1,\mu_0)=\frac{\varpi}{2}\int_0^{\tau_1}E_1(|t+\tau-u|)J(n,u,\tau_1,\mu_0)du$$

$$=\int_0^{\tau}E_1(t+\tau-u)J(n,u,\tau_1,\mu_0)du+\frac{\varpi}{2}\int_{\tau}^{\tau_1}E_1(|t+\tau-u|)J(n,u,\tau_1,\mu_0)du. \quad (3.3)$$

Inserting equation (2.9) into the first term of the last hand side of equation (3.3), we have

$$J(n+1,t+\tau,\tau_1,\mu_0)=\frac{\varpi}{2}\int_0^{\tau}\int_0^1 e^{-(t+\tau-u)/\mu}J(n,u,\tau_1,\mu_0)du$$

$$+\frac{\varpi}{2}\int_0^{\tau_1-\tau}E_1(|t-v|)J(n,v+\tau,\tau_1,\mu_0)dv. \quad (3.4)$$

By the assumption of induction, we can substitute equation (3.1) into the second term of the right hand side of equation (3.4), hence we find

$$J(n+1,t+\tau,\tau_1,\mu_0)=\frac{\varpi}{2}\int_0^1 e^{-t/\mu}\frac{d\mu}{\mu}\int_0^{\tau}J(n,u,\tau_1,\mu_0)e^{-(\tau-u)/\mu}du$$

$$+e^{-\tau/\mu_0}\frac{\varpi}{2}\int_0^{\tau_1-\tau}E_1(|t-v|)J(n,v,\tau_1-\tau,\mu_0)dv$$

$$+\frac{2}{F}\sum_{k=1}^{n-1}\int_0^1 I(n-k,\tau,\mu')d\mu'\frac{\varpi}{2}\int_0^{\tau_1-\tau}E_1(|t-v|)J(k.v,\tau_1-\tau,\mu')dv. \quad (3.5)$$

Substituting from equation (2.6) and recalling equations (2.3) and (2.10),we obtain equation (2.1) for $n+1$.

Now we treat equation (3.2). For $n=1$ this is valid since

$$J(1,t,\tau_1,\mu_0)=J(1,t,\tau,\mu_0)=\frac{\varpi F}{4}e^{-t/\mu_0}$$

Suppose equation (3.2) is true up to n. From equation (2.3) we have

$$J(n+1,t,\tau_1,\mu_0)==\frac{\varpi}{2}\int_0^{\tau}J(n,u,\tau_1,\mu_0)E_1(|u-t|)du$$

$$+\frac{\varpi}{2}\int_{\tau}^{\tau_1}J(n,u,\tau_1,\mu_0)du\int_0^1 e^{-(u-t)/\mu}\frac{d\mu}{\mu}. \quad (3.6)$$

The first term of the right hand side of equation (3.6) can be substituted from equation (3.2) because of the assumption of induction. Taking account of equations (2.3) and (2.9), we have

$$J(n+1,t,\tau_1,\mu_0)=J(n+1,t,\tau,\mu_0)$$

$$+\frac{2}{F}\sum_{k=1}^{n-1}\int_0^1 I(n-k,\tau,-\mu')d\mu'\frac{\varpi}{2}\int_0^{\tau} J(k,\tau-u,\tau,\mu_0)du$$

$$+\frac{2}{F}\int_0^1 \frac{\varpi F}{4}e^{-(\tau-t)/\mu'}d\mu'\int_{\tau}^{\tau_1} J(n,u,\tau_1,\mu_0)e^{-(u-\tau)/\mu'}\frac{du}{\mu'}$$

$$=J(n+1,t,\tau,\mu_0)+\frac{2}{F}\sum_{k=1}^{n-1}\int_0^1 I(n-k,\tau,-\mu')J(k+1,\tau-t,\tau,\mu')d\mu'$$

$$+\frac{2}{F}\int_0^1 J(1,\tau-t,\tau,\mu')I(n,\tau,-\mu')d\mu'.$$

Therefore we obtain that equation (3.2) for $n+1$ is valid. This completes the proof. □

4. The Principles of Invariance

With the aid of functional relations satisfied by the finite order source function just proved in the previous section, we shall derive the principles of invariance for finite order reflection and transmission functions.

Multiplying equation (3.1) by $e^{-t/\mu}/\mu$,integrating with respect to t,over $(0,\tau_1-\tau)$, and taking account of equation (2.13), we have

$$I(n,\tau,-\mu)=\frac{F}{4\mu}e^{-\tau/\mu_0}S(n,\tau_1-\tau,\mu,\mu_0)$$

$$+\frac{1}{2\mu}\sum_{k=1}^{n-1}\int_0^1 S(k,\tau_1-\tau,\mu,\mu')I(n-k,\tau,+\mu')d\mu'. \tag{4.1}$$

This corresponds to principle i of Chandrasekhar(1960), which is stated by Bellman, Ueno and Vasudevan (1972). After multiplication equation (3.1) by $\exp[-(\tau_1-\tau-t)/\mu]$ and integration with respect to t over $(0,\tau_1-\tau)$, the left hand side becomes

$$\int_0^{\tau_1-\tau} J(n,t+\tau,\tau_1,\mu_0)e^{-(\tau_1-\tau-t)/\mu}\frac{dt}{\mu}$$

$$=e^{-\tau/\mu_0}\int_0^{\tau_1-\tau} J(n,t,\tau_1-\tau,\mu_0)e^{-(\tau_1-\tau-t)/\mu}\frac{dt}{\mu}$$

$$+\frac{2}{F}\sum_{k=1}^{n-1}\int_0^1 I(n-k,\tau,+\mu')d\mu'\int_0^{\tau_1-\tau} J(k,t,\tau_1-\tau,\mu')e^{-(\tau_-\tau-t)/\mu}\frac{dt}{\mu}. \tag{4.2}$$

The left hand side of equation (4.2) reads

$$\int_0^{\tau_1} J(n,\tau_1,\mu_0)e^{-(\tau_1-t)/\mu}-\int_0^{\tau} J(n,t,\tau_1,\mu_0)e^{-(\tau-t)/\mu}\frac{dt}{\mu}.$$

Recalling equations (2.6) and (2.14), the left hand side of equation (4.2) becomes

$$\frac{F}{4\mu}T(n,\tau_1,\mu,\mu_0)-e^{-(\tau_1-\tau)/\mu}I(n,\tau,+\mu). \tag{4.3}$$

The first term of the right hand side of equation (4.2) reads

$$\frac{F}{4\mu}e^{-\tau/\mu_0}T(\tau_1-\tau,\mu,\mu_0). \tag{4.4}$$

The second term of the right hand side of equation (4.2) becomes (also by equations (2.6) and (2.14))

$$\frac{1}{2\mu}\sum_{k=1}^{n-1}\int_0^1 T(k,\tau_1-\tau,\mu,\mu')d\mu'. \tag{4.5}$$

From equation (4.2) with equations (4.3) through (4.5), we obtain

$$\frac{F}{4\mu}T(n,\tau_1,\mu,\mu_0)=\frac{F}{4\mu}e^{-\tau/\mu_0}+e^{-(\tau_1-\tau)/\mu}I(n,\tau,+\mu)$$

$$+\frac{1}{2\mu}\sum_{k=1}^{n-1}\int_0^1 T(k,\tau_1-\tau,\mu,\mu')I(n-k,\tau,+\mu')d\mu'. \tag{4.6}$$

This corresponds to principle iv of Chandrasekhar(1960). Taking account of polarization, Fymat and Ueno (1974) stated the same equation. Integration of equation (3.2) multiplied by $e^{-(\tau-t)/\mu}/\mu$ with respect to t over $(0,\tau)$ yields the following equation, which corresponds to principle ii of Chandrasekhar (1960):

$$I(n,\tau,+\mu)=\frac{F}{4\mu}T(n,t,\tau,\mu_0)$$

$$+\frac{1}{2\mu}\sum_{k=1}^{n-1}\int_0^1 S(k,\tau,\mu,\mu')I(n-k,\tau,-\mu')d\mu', \tag{4.7}$$

where equations (2.6), (2.13) and (2.14) are taken into account. Bellman, Ueno and Vasudevan (1972) stated equation (4.7). In a similar manner used in driving equation (4.6), multiplying equation (3.2) by $e^{-t/\mu}$ and integrating with respect to t over $(0,\tau)$, we obtain

$$\frac{F}{4\mu}S(n,\tau_1,\mu,\mu_0)=\frac{F}{4\mu}S(n,\tau,\mu,\mu_0)$$

$$+e^{-\tau/\mu}I(n,\tau,-\mu)+\frac{1}{2\mu}\sum_{k=1}^{n-1}\int_0^1 T(k,\tau,\mu,\mu')I(n-k,\tau,-\mu')d\mu', \tag{4.8}$$

where equations (2.5), (2.13) and (2.14) are used. This corresponds principle iii of Chandrasekhar (1960).

Equations (4.1), (4.7), (4.6) and (4.8) constitute a complete set of principles of invariance for finite order reflection and transmission functions. On summing over all n with taking account of the following relation

$$\sum_{n=2}^{\infty}\sum_{k=1}^{n-1} a_k b_{n-k} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_n b_k$$

under the assumption that $\sum a_n$ and $\sum b_n$ converge, we have the complete set of principles of invariance stated by Chandrasekhar (1960) as follows:

$$I(\tau,\mu) = \frac{F}{4\mu} e^{-\tau/\mu} S(\tau_1 - \tau, \mu, \mu_0) + \frac{1}{2\mu}\int_0^1 S(\tau_1 - \tau, \mu, \mu')I(\tau, +\mu')d\mu', \tag{4.9}$$

$$I(\tau,+\mu) = \frac{F}{4\mu} T(\tau, \mu, \mu_0) + \frac{1}{2\mu}\int_0^1 S(\tau, \mu, \mu')I(\tau, -\mu')d\mu', \tag{4.10}$$

$$\frac{F}{4\mu} S(\tau_1, \mu, \mu_0) = \frac{F}{4\mu} S(\tau, \mu, \mu_0) + e^{-\tau/\mu} I(\tau, -\mu) + \frac{1}{2\mu}\int_0^1 T(\tau, \mu, \mu')I(\tau, -\mu')d\mu', \tag{4.11}$$

and

$$\frac{F}{4\mu} T(\tau_1, \mu, \mu_0) = \frac{F}{4\mu} e^{-\tau/\mu_0} T(\tau_1 - \tau, \mu, \mu_0) + e^{-(\tau_1-\tau)/\mu} I(\tau, +\mu) + \frac{1}{2\mu}\int_0^1 T(\tau_1 - \tau, -\mu')d\mu'. \tag{4.12}$$

5. Concluding Remarks

Starting with the equation of transfer satisfied a specific intensity scattered isotropically n times in a homogeneous atmosphere of finite optical thickness, we exactly obtain a set of functional relations satisfied by n-th order source function (equations (3.1) and (3.2)). With the aid of the functional relations the principle of invariance satisfied the reflection and transmission functions of n-th order follow (equations (4.1), (4.7), (4.8) and (4.6)).

In driving these equations, we frequently used the inversion of the order of integrations (cf. equations (3.4) and (3.5)), which is justified by the fact that all integrands are non negative, in the sense that either all integrals are finite and equal or all are infinite.

The verifications of the finite order principles of invariance for the cases including anisotropic scattering and polarization will be discussed in forthcoming papers.

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