Solution of Vlasov Equation II

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Abstract

The linear perturbation of the Vlasov equation is solved in the case of electromagnetic perturbations in a general coordinate system. The adiabatic component of the solution has a new term induced from the diamagnetic effect. The kinetic solution is found to reduce the electromagnetic gyrokinetic solution in the low frequency limit. Some characteristics of non−abiabatic component of solution are examined for both circulating and trapped particles.

§ 1. Introduction

In a previous report [1], the linear perturbation of Vlasov equation has been solved by introducing a propagator in a general coordinate system in the electrostatic approximation. Inrealistic comfined plasmas, the magnetic perturbations become incleasingly important for various plasma stabilities and plasma transport. In this report, we develop the method to solve the linear perturbation of Vlasov equation in the case of electromagnetic perturbations by applying the propagator used in the electrostatic case [1].

§ 2. Kinetic Solution with Electromagnetic Perturbations

We begin with the Vlasov equation for plasma distribution function $f(x, v, t)$

$$\left(\frac{\partial}{\partial t}+L\right)f(x, v, t)=0, \qquad \cdots\cdots\cdots(1)$$

where the operator L has been defined in the previous report [1]. When it is expressed in the form, $L=\bar{L}+\tilde{L}$, the ensemble averaged part $\bar{L}$ is the same as Eq. (5) in Ref. 1) .

In the case of electromagnetic perturbations,the perturbed portion has magnetic perturbations:

$$\tilde{L}=\frac{e}{m}\left(-\nabla\tilde{\phi}+\frac{i\omega}{c}\underline{\tilde{A}}+\frac{1}{c}\underline{V}\times\underline{\tilde{B}}\right)\frac{\partial}{\partial v}, \qquad \cdots\cdots\cdots(2)$$

where $\tilde{B}=\nabla\times\tilde{A}$ is the magnetic perturbation and $\tilde{A}$ is the perturbed vector potential. We will use the same notations as used in Ref. [1]. With this operator $\tilde{L}$,the linear perturbation of Eq.(1) can be written by

$$\tilde{f} = - T \tilde{L} \bar{f} . \qquad \cdots\cdots(3)$$

where the solution operator T ,Which is equivalent to the propagator,is defined by

$$T = \int_{-\infty}^{t} dt' \exp\left(\int dt'' \bar{L}\right). \qquad \cdots\cdots(4)$$

The unperturbed equilibrium solution $\bar{f}$ is given by the same form as given by Eq.(10) in Ref.(1). Making use of the eikonal representation for the perturbation: $\tilde{f} = f \exp(-i\omega t + iS(x))$, and decomposing A in the form[2]

$$A = \underline{b}\psi + i\gamma\nabla S - i\sigma\underline{b}\times\nabla s , \qquad \cdots\cdots(5)$$

we have the perturbed magnetic field in terms of ψ, γ and σ:

$$\underline{B} = \nabla\psi\times\underline{b} + i\nabla\delta\times\nabla S - i\nabla\sigma\times(\underline{b}\times\nabla S). \qquad \cdots\cdots(6)$$

The cross product of Eq.(6) with $\underline{v} = \underline{v}_{11} b + v_{\perp}$ yields

$$\underline{v}\times\tilde{\underline{B}} = v_{11}\nabla\psi - (\underline{v}_{\perp},\nabla\psi)\underline{b} + i(\underline{v},\nabla S)\nabla\gamma - i(\underline{v},\nabla\gamma)\nabla S$$
$$- i(\underline{v},\underline{b}\times\nabla S)\nabla\sigma + i(\underline{v},\nabla\sigma)\underline{b}\times\nabla S. \qquad \cdots\cdots(7)$$

Introducing Eqs.(5),(6) and (7) into Eq.(2),we have the source term for Eq.(3) in the electromagnetic perturbations:

$$\tilde{L}\bar{f} = \frac{e}{m}\Big[-(\underline{\nabla},\nabla\tilde{\phi})\left(\frac{\partial F}{\partial\varepsilon}+\frac{1}{B}\frac{\partial F}{\partial\mu}\right) - \frac{1}{B}\frac{\partial F}{\partial\mu}(v_{11}\underline{b}\nabla + \underline{v}_d\nabla)\tilde{\phi}$$
$$+ \frac{\nabla\tilde{\phi}\,\underline{V}\times\underline{b}}{v_1{}^2\Omega}(\underline{V}_1,\nabla F) + \frac{i}{c}\omega\Big\{\tilde{\underline{A}},\underline{V}\ \frac{\partial F}{\partial\varepsilon}+\frac{1}{B}\frac{\partial F}{\partial\mu}$$
$$-\frac{1}{B}\frac{\partial F}{\partial\mu}\tilde{\underline{A}}(v_{11}\underline{b}+\underline{v}d) + \frac{\tilde{A},\underline{V}\times b}{v_{\perp}{}^2\Omega}(v_{\perp},\nabla F)\Big\}$$
$$-\frac{1}{CB}\frac{\partial F}{\partial\mu}\underline{V}\times\tilde{\underline{B}}(v_{11}\underline{b}+\underline{v}d) - \frac{(\underline{V}\times\tilde{\underline{B}})\nabla\times\underline{b}}{ev_{\perp}{}^2\Omega}(\underline{V}_1,\nabla F)\Big]. \qquad \cdots\cdots(8)$$

Equation (8) without magnetic perturbations $\tilde{\underline{A}}$ and $\tilde{\underline{B}}$, i , e .,the first,second and third terms reduce to the electrostatic result (Eq.(16) in Ref.(1)) .

Making use of the relations: $\underline{A}.\underline{V} = v_{11}\psi + i\gamma\underline{V}.\nabla S - i\sigma(\underline{b}\times\nabla S)\underline{V}$, $\underline{A}.\underline{b} = \psi$, $\underline{A}.\underline{V}_d = i\gamma\omega_d - i\sigma(b\times\nabla S),\underline{V}_d$, $\underline{A}.\underline{V}\times\underline{b} = -i\gamma(\nabla S\times\underline{b}),\underline{V} + i\sigma(\underline{V},\nabla S)$ and the formula for the operator T:

$$T(\underline{V},\nabla)\tilde{\phi} = \tilde{\phi} - T\frac{\partial\tilde{\phi}}{\partial t}, \qquad \cdots\cdots(9)$$

we obtain,from Eqs.(3) and (4),a representation for $\tilde{f}$:

$$\tilde{f} = \frac{e}{m}\left[\left(\frac{\partial F}{\partial\varepsilon}+\frac{1}{B}\frac{\partial F}{\partial\mu}\right)\tilde{\phi} - \frac{v_{11}}{CB}\tilde{\psi}\frac{\partial F}{\partial\mu} + U\right]$$
$$-\frac{e}{m}T\left[\left\{i(\omega-\omega d + i v_{11} b\nabla)\frac{1}{B}\frac{\partial F}{\partial\mu} - iQ\right\}V + i\omega U\right], \qquad \cdots\cdots(10)$$

where Q, U and V are defined as follows: $Q = \omega \partial F / \partial \varepsilon - (\nabla S \times \underline{b})(\underline{v}_{\perp} \times \nabla F) / (\Omega v_{\perp}^2)$,

$$U = i \frac{(v_{\perp}, \nabla F)}{c \Omega v_{\perp}^2} (\nabla S, v \times b \gamma - (b \times \nabla S)(v \times b) \hat{\sigma}), \quad \cdots(11)$$

$$V = \hat{\phi} - \frac{v_{11}}{c} \hat{\psi} + \frac{\hat{\sigma}}{c} \underline{V} . (\underline{b} \times \nabla S) - \frac{\hat{\gamma}}{c} (\underline{v} . \nabla S). \quad \cdots\cdots\cdots\cdots\cdots(12)$$

The terms in the first square brackets in Eq.(10) are adiabatic terms which is independent of ω. The quantity U, which is new, is induced from the perpendicular components of the vector potential and the diamagnetic effect.

We now consider some characteristics of the propagator T in a toroidal geometry taking into account the effect of global toroidal gyromotion[3] for circulating particles. In an axisymmetric tokmak, the eikonal function has been given by [4] $S = -n(\theta - \int q\, dx)$. Even in the zxisymmetric toroidal surface, If we go along an magnetic field line on the toroidal surface, the symmetry is broken down. For example the pitch length of the field lined is shorter in the inside than that in the outside of the torus. The safety factor in this case may be written as $q = q_0(1 - \varepsilon(1 + \Lambda)\cos x)$, and the eikonal function becomes $S = -n(\theta - \int q_0\, dx + a_m \cos x)$, where q_0 is the safety fuctor in the straight cylindrical geometry, and $a_m = \varepsilon m (1 + \Lambda)$. The second oscillating term in S represents the global toroidal gyromotion of the field lines. If we take into account this toroidal gyromotion, together with the effect of small scale Larmor gyromotion, the propagator may be written in the form

$$T = \sum_{p l p'} J_p(a_m) J_{p'}(a_m) \sum_{k l k'} \frac{J_k(a) J_{k'}(a)}{\omega - \omega_d - i v_{11} b \nabla - k \Omega} e^{i(k-k')\zeta} e^{i(p-p')x} \quad \cdots\cdots\cdots\cdots\cdots(13)$$

where $J_p(a_m)$ represent the effect of global toroidal gyromotion, while $J_k(a)$ represents the effect of small scale Larmor gyromotion. Without the toroidal effect, $a_m \to 0$, Eq.(13) reduces to one derived in Ref.(1). For the sake of simplicity, we will neglect $J_p(a_m)$ in the followings.

Substitution of Eq.(13) without J into Eq.(10), we have

$$f = \frac{e}{m} [\phi \frac{\partial F}{\partial E} + \frac{1}{B} \frac{\partial F}{\partial \mu} - \frac{v_{11}}{c} \psi \frac{1}{B} \frac{\partial F}{\partial \mu} + U + \sum_{k l k'} J_k J_{k'} e^{i(k-k')\zeta} \frac{V}{B} \frac{\partial F}{\partial \mu}$$
$$+ \sum_{k l k'} J_k J_{k'} e^{i(k-k')\zeta} (\omega - \omega_d - i v_{11} b . \nabla - k \Omega)^{-1}$$
$$\{(k \frac{\Omega}{B} \frac{\partial F}{\partial \mu} + Q) V + i \omega U\}] \quad \cdots\cdots\cdots\cdots\cdots(14)$$

§ 3. Gyrokinetic Solution

If we average Eq.(14) over the gyrophase angle S for the low frequency limit $\omega \ll \Omega$, taking only $k = 0$ in Eq.(14), we have

$$f=\frac{e}{m}\quad\phi\quad\frac{\partial F}{\partial\varepsilon}+\frac{1}{B}\ \frac{\partial F}{\partial\mu}\quad-\frac{v_{11}}{C}\phi\frac{1}{B}\ \frac{\partial F}{\partial\mu}+U+J_0(a)(V-h)\ , \qquad \cdots\cdots(15)$$

$$h=(\omega-\omega d-i\ v_{11}\underline{b}.\nabla)^{-1}\bar{Q}\ \bar{V}, \qquad \cdots\cdots(16)$$

The averaged quantities are given as follows: $\bar{Q}=\omega\ \partial F/\partial\varepsilon-\nabla S\times\underline{b}.\nabla F/\Omega$,

$V\quad(\phi\quad\frac{v_{11}}{C}\phi)\ J_0(a)+\frac{v_\perp}{c}\quad\nabla S\quad(\sigma+\gamma)\ J_1(a)$,

$U\quad\frac{(\gamma+\sigma)}{2\ c\ \Omega}(\nabla F\nabla S)$.

Equation(15)is the gyrokinetic solution for electromagnetic perturbations.As compared with the gyrokinetic solution given by Antonsen et al[2],we have the diamagnetic contri－bution U,and the contribution from γ in $\bar{U}$ and $\bar{V}$. The function h given by Eq.(16) correspnds to the h in Ref.(2),which was derived iteratively from the Vlasov equation in entirely different manner.

Here we will examine some characteristics of the solution h which plays an import－ant role in plasma kinetic theory. The inverse differential operator can be transformed to an integral operator by the rule:

$$(P+\frac{d}{d\ l})^{-1}=e^{-\int P d l}\int dl'\ e^{\int P d l'}. \qquad \cdots\cdots(17)$$

Applying this formula (17) for Eq.(16) with $\underline{b}.\nabla=d/d\ l$,we have representations $h^\pm$ corresponding to $v_{11}\gtrless 0$:

$$h^\pm(x)=e^{\pm i I(x,x')}\ h^\pm(x_0)\pm\int dx'\ e^{\pm i I(x,x')}\ (A\pm B), \qquad \cdots\cdots(18)$$

where the field line length l is transformed to the poloidal angle－like variable x, $I(x,x')=\int Rq(\omega-\omega_d)/|v_{11}|dx$, $A=RqQ(J_0(a)\phi+J_1(a)|\nabla S|(\sigma+\gamma)$ $c)\quad v_{11}$ and $B=qRQJ_0(a)\phi/c$.

For passing particles,the periodicity condition $h^\pm(x_0+2\pi)=h^\pm(x_0)$ yields the sum

$$h^-+h^-=\frac{1}{\sin(I_0/2)}\oint dx\ \{iAK_a(x,x')+BK_b(x,x')\} \qquad \cdots\cdots(19)$$

where the kernels K_a and K_b have been given by

$$K_a=\begin{cases}\cos(I(x,x')+I_0/2)\\ \cos(I(x,x')-I_0/2)\end{cases}$$

$$K_b=\begin{cases}\sin(I(x,x')+I_0/2)\ \text{for}\ x'>x,\\ \sin(I(x,x')-I_0/2)\ \text{for}\ x'<x.\end{cases} \qquad \cdots\cdots(20)$$

The difference is written in the form

$$h^-+h^-=\frac{1}{\sin(I_0/2)}\oint dx'\ \{Ak_b(x,x')+iBK_a(x,x')\} \qquad \cdots\cdots(21)$$

where $I_0=I(2\pi,0)=2\pi(\omega-\omega_d)/\omega_t$ and the average for the circulating parti－cles is defined by

$$\overline{\omega}_d = \oint \frac{dl}{|v_{11}|}\omega_d \Big/ \oint \frac{dl}{|v_{11}|} \quad \cdots\cdots\cdots(22)$$

The sum given by Eq.(19) is used for the even moment integral with respect to v_{11} such as the perturbed density,while the difference given by Eq.(21) is used for the odd moment integral such as the evaluation of perturbed current. At the resonance condition, $\sin(I_0/2)=0$ or $I_0=2\pi p$,Eqs. (19) and (21) have the singularity. At this resonance condition,the kernels become continuous:$K_a(x,x')=(-1)^p\cos I(x,x')$ and $K_p(x,x')=(-1)^p\sin I(x,x')$.In the fluid limit, $a<<1$,in particular,the solution (19) is simplified:

$$h^+ + h^- = \frac{2i\omega_t}{\sin(I_0/2)}\overline{Q\cos I(x,x')\frac{\omega_d}{\omega}\phi} \quad \cdots\cdots\cdots\cdots\cdots(23)$$

We now derive the solution for trapped particles by imposing the continuity condition at the turning points $x=x_{\pm}$: $h^+(x_+)=h^-(x_+)$ and $h^+(x_-)=h^-(x_-)$. With these conditions, we have the solution

$$h^+ + h^- = \frac{2}{\sin I_1}\int_{x_-}^{x_+} dx' \{iAK_1(x,x')-BK_2(x,x')\} \quad \cdots\cdots\cdots\cdots\cdots(24)$$

where the kernels K_1 and K_2 have been given by [5]

$$K_1=\begin{cases}\cos I(x',x_+)\cos I(x,x_-) \\ \cos I(x',x_-)\cos I(x_+,x)\end{cases}, \quad K_2=\begin{cases}\sin I(x',x_+)\cos I(x,x_-) & \text{for } x'>x, \\ \sin I(x',x_-)\cos I(x,x_+) & \text{for } x'<x,\end{cases}$$

and $I_1=I(x_+,x_-)=2\pi(\omega-\overline{\omega}_d)/\omega_b$ with $\omega_b=2\pi/\oint dl/|v_{11}|$. The difference can be obtained by the same manner making use of the formulae:

$$\sin I(x',x_-)\sin I(x_+,x)+\sin I(x,x')\sin I_1=\sin I(x_+,x')\sin I(x,x_-),$$

$$\cos I(x,x')\sin I_1-\cos I(x',x_-)\sin I(x_+,x)=\cos I(x_+,x')\sin I(x,x_-),$$

in the form

$$h^+ + h^- = \frac{2}{\sin I_1}\int_{x_-}^{x_+} dx' \{AK_3(x,x')+iBK_4(x,x')\},$$

where the kernels K_3 and K_4 have been defined by

$$K_3=\begin{cases}\cos I(x_+,x')\sin I(x_-,x) \\ \cos I(x',x_-)\sin I(x_+,x)\end{cases}, \quad K_4=\begin{cases}\sin I(x_+,x')\sin I(x,x_-) & \text{for } x'>x, \\ \sin I(x_+,x)\sin I(x',x_-) & \text{for } x'<x,\end{cases}$$

At the resonance condition, $\sin I_1=0$,these kernels K_1 and K_2 become continuous as in the case of circulating particles. Near the resonance, $I_1=\pi p$, in the fluid limit, $a<<1$,the solution (24) is simplified in the form

$$h^+ + h^- = 2\frac{\pi\cos I(x,x_+)}{\sin I_1\omega_b}\overline{Q\frac{\omega_d}{\omega}\cos I(x,x_-)\phi} \quad \cdots\cdots\cdots\cdots(25)$$

At the resonance condition, $I_1=I(x_+,x_-)=\pi p$,function $I(x,x_-)$ varies form 0

to πp as x varies from x_- to x_+. Since $\partial I(x, x_-)/\partial x = Rq(\omega - \bar{\omega}_{11})/|v_{11}|$, from the definition, $\partial I/\partial x$ tends to infinity at the edges $x = x_\pm$. This means that $I(x, x_-)$ increases very sharply near the edges. $x = x_\pm$. For $p = 0$, $I(x, x_-) << \pi/2$ and we may approximate by $\overline{\cos I(x, x_-)} \sim 1$.
For $p \neq 0$,from the definition of $I(x, x_-)$,we have

$$I(x, x_-) = \frac{p\omega_b}{\omega_t} \cdot \frac{2}{(2\varepsilon\lambda_0)y_2}\{K(k) - F(k, \alpha)\},$$

where $p\omega_b >> \bar{\omega}_d$ has been assumed, $\lambda_0 = \varepsilon/\mu$, K and F are the first kind complete and incomplete elliptic functions,respectively, $k^2 = (1 - \lambda_0 + \varepsilon\lambda_0)/(2\varepsilon\lambda_0)$ and $\alpha = \sin^{-1}(k^{-1}\sin x/2)$.

For $p = 1$,the function $\cos I(x, x_-)$ becomes odd with respect to x and its integral over (x_-, x_+) vanishes. Therefore,we have no $p = 1$ resonance contribution in Eq.(25). For $p = 2$ resonance,$\cos I(x, x_-)$ becomes even,and the average integral in Eq.(25),in this case,does not vanishes, i. e.,the second harmonic resonance may have some contribution in Eq.(25).

§ 4. Summary

We have solved the linear perturbation of Vlasov equation with the full electromagnetic perturbations in a general coordinate system. The adiabatic part of the kinetic solution has a new term which is induced from the perpendicular components of perturvedvector potential and the diamagnetic effect. When averaged over the gyrophase angle in the low frequency regime, the kinetic solution is found to reduce the electromagnetic gyrokinetic solution. Approximate forms of the gyrokinetic solution have also derived for both circulating and trapped particles.

References

1) T. Yamagishi,Memoirs of Fukui Institute of Technology, No.19 (1989) 365.
2) T. M. Antonsen,Jr. and B. Lane, Phys. Fluids, 23 (1980) 1205.
3) T. Yamagishi,J. Phys. Soc. Japan, 57 (1988) 2732.
4) J. W. Conner,R.J.Hastie, and J.B.Taylor, Proc. Royal Soc. London Ser. A 365 (1979) 1.
5) T. Yamagishi,J. Phys. Soc. Japan,59 (1990) 138.