Correction to My Paper

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1. Introduction

In the paper of preceding FIT Memoirs I asserted that, if R is an artinian ring with $J(R)^2 = 0$ and S is a simple non-projective right R-module, then the dual module S^* is simple or zero. This assertion is false. The dual module is semisimple but not necessarily simple, in other words, there is an artinian ring R with $J(R)^2 = 0$, and there is a simple non-projective right R-module S the dual module of which is a direct sum of more than one simple module. Therefore the other part of the paper is not valid without the condition:

The dual module of simple non-projective right R-module is simple or zero,

where the square of the radical of R is zero.

In this paper 'ring' is assumed to have an identity element, 'module' is assumed to be unitary, and the radical of a ring R is denoted by $J(R)$.

We study the condition over artinian ring R with $J(R)^2 = 0$ from a view point of homological algebra.

2. Semisimplicity

A non-zero module M is called simple if zero is the only proper submodule of M and a module is called semisimple if it is a direct sum of a finite number of simple modules. A submodule N of a module M is called maximal if the factor module M/N of M by N is simple. The intersection of all maximal submodules of M is called the (Jacobson) radical of M and denoted by $J(M)$. A ring R is considered to be a right or left R-module by the multiplication of the elements of R on the right or left respectively, this right module is denoted by R_R and this left module is denoted by ${}_RR$. The radical

$J(R_R)$ and $J({}_RR)$ coincide and are denoted by $J(R)$. The radical plays an important role in ring theory.

If there is a chain $0=M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n=M$ of a module M with simple M_i/M_{i-1} $(i=1,...,n)$, the number n is called the length of M. This does not depend on the choice of the chain. The length of a submodule of a module of finite length is finite, and the length of a factor module of a module of finite length is finite. A right R-module M of finite length is semisimple if and only if $MJ(R) = 0$.

The set $\mathrm{Hom}(M,R_R)$ of all homomorphisms from a right R-module M to a ring R (considered right R-module) has a structure of left R-module by $(rf)(x) = r(f(x))$, where r and x are elements of R, and f is a homomorphism. This module of homomorphisms is also denoted by M^* for brevity and is called the dual module of M. If M is a non-projective indecomposable module, then any homomorphic image $f(M)$ of M is in $J(R)$. Therefore, if $J(R)^2 = 0$, the dual module of a simple non-projective module is semisimple. The following example shows that this dual module is not always simple.

Example. Let K be a field, $K \oplus K$ be the direct sum of two copies of K as K-K-bimodule. Let R be the matrix ring

$$\begin{bmatrix} K & K\oplus K & 0 \\ 0 & K & K \\ 0 & 0 & K \end{bmatrix}$$

P be the second row part $[\ 0 \quad K \quad K\ ]$ of R and Q be the submodule $[\ 0 \quad 0 \quad K\ ]$ of P. Then the dual module of P/Q is a direct sum of two simple modules.

3. Classification of simple modules

We classify simple modules over an artinian ring R with $J(R)^2 = 0$ from a homological view point.

A module M is called projective if for any epimorphism $f: X \to Y$ and any homomorphism $g: M \to Y$ there exists a homomorphism $h: M \to Y$ such that $g = fh$. Dually, a module M is called injective if for any monomorphism $f: X \to Y$ and any homomorphism $g: X \to M$ there exists a homomorphism $h: Y \to M$ such that $g = hf$.

The simple R-modules are classified nominally in four classes: the class of projective injective modules, the class of projective non-injective modules, the class of non-projective injective modules and the class of non-projective non-injective modules. But if there exists a projective injective simple module S, then the ring R is decomposed as $R = R_1 \times R_2$, where R_1 is the sum of all homomorphic images of S and R_1 is semiprime, and we can concentrate our attention on R_2 a ring over which there is no projective injective simple module.

3. Dual modules

The type of a simple module is characterized by the dual of the simple module.

Theorem: Let R be a non-semisimple artinian ring with $J(R)^2 = 0$ which is indecomposable as ring, and let S be a simple right R-module.

(1) If S is projective and non-injective, then S^* is a projective non-simple module($\neq 0$).

(2) If S is injective and non-projective, then $S^* = 0$.

(3) If S is non-projective and non-injective, then S^* is a non-zero semisimple module.

Proof. (1) There is a direct summand of R isomorphic to S, and $S^* \neq 0$. Suppose that $S^* = 0$ and P is a projective module not isomorphic to S. Then $\mathrm{Hom}(P,S) = 0$ and $\mathrm{Hom}(S,P) = 0$. Hence the sum R_1 of all homomorphic images of S is a semisimple ring and induces a ring decompositon $R = R_1 \times R_2$. This contradicts the assumption.

(2) If $S^* \neq 0$, then there is a non-zero homomorphism from S to R, which is monomorphism. Since S is injective, there exists a direct summand of R isomorphic to S and S is projective, which is a contradiction.

(3) If S is non-injective, there is a non-simple local module L (e.g. summand of the second socle of the injective hull of S) such that S is subisomorphic to L. Since $J(R)^2 = 0$, the simple module S is subisomorphic to the projective cover of L. Therefore $S^* \neq 0$.

The following lemma is obtained immediately from this lemma.

Lemma: Let R be a non-semisimple artinian ring with $J(R)^2 = 0$ which is indecomposable as ring, and let S be a simple right R-module.

(1) If S^* is a non-simple projective module, then S is a projective non-injective simple module.

(2) If $S^* = 0$, then S is injective.

(3) If S^* is semisimple non-zero module, then S is a non-projective non-injective module.

REFERENCES

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(平成2年12月20日 受理)