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剛体球フェルミ粒子系のエネルギーに対する励起粒子の自己エネルギーの影響
－三体相関の効果－

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Three-Particle Correlation and the Off Energy-Shell Self-Energy of Excited Particles in a Fermion System of Hard Spheres

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Abstract

The energy shift in the ground state system of hard sphere fermions is studied with particular emphasis on the off energy-shell self-energy effect of excited particles induced by three-particle correlation. The energy shift is known to diverge when one attempts to analyze it in terms of K- or G-matrix which describes the two-particle correlation successfully. Use is made of a new method which deals with the hard sphere interaction in terms of effective potential. The energy shift of interest is expressed as a convergent integral which varies as $a^4 \log a$ for small values of the sphere radius a. The contribution to the integral of order a^4 is also identified.

Introduction

Strong repulsion between the particles at close distance in quantum fluids prevents direct application of the linked cluster expansion method which offers the basis for theoretical study of quantum many-particle systems.[1] The K-matrix and G-matrix methods which have been developed for the study of fermion systems avoid this difficulty by taking account of repeated collisions of two particles.[2][3] They are based on the infinite partial summation of perturbation terms in the energy and other quantities. They can give reasonable results in a way similar to the case of two-particle system in free space or scattering system. In fact, however, the infinite summation method does not carry out infinite sum in the true sense of the word. Its basic assumption is that the perturbation series for energy, for example, is analytic at the vanishing value of the potential strength, and that the series can be continued analytically from the weakest potential to the actual potential of interest.

On the other hand, the nonanalyticity of the whole perturbation series at the vanishing

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value of the potential strength has been known for long years. The series can at best be only an asymptotic series.[4)5)] Thus one has to be content with the asymptotic approximation confining himself to weakest interaction, if he respects the whole series. Or one can only hope that nonanalytic part of the whole sum to be sufficiently small and that the main part admits analytical continuation. The nonanalyicity is expected because the convergence of two series for interactions of equal strengths but of opposite signs are governed by the same radius of convergence, while a large system would collapse if the interaction is attractive. Baker attributed the vanishing radius of convergence in nonrelativistic many-body problems to the fact that the number of perturbation terms increases quite rapidly — like combinatorial factors — as one proceeds to higher orders.

One might expect that the interaction of three particles in a fermion system can be described as a combined effect of two-particle correlation described by K- or G- matrix as far as the particles interact via two-body force. Attempts along this line of thought have not been successful, and one is met with ultraviolet divergence in the calculation of three-particle correlation energy.[6)7)] In particular, Efimov has shown that even a mild potential leads to divergence. From the divergent behavior of some integrals (the precursor terms) representing the three-particle correlation energy, he expected the energy would involve terms of order $(k_0 a_s)^4 \log(k_0 a_s)$ in dimensionless scale, where k_0 is the Fermi wave number, and a_s is the s-wave scattering length. As a_s, which may serve as a substitute for hard sphere radius, is an analytic function of the potential strength, Efimov was led to nonanalytic result by assuming the possibility of analytical continuation when he introduced cutoffs to divergent energy integrals. Note that, as far as one confines himself to the correlation of three particles, the number of higher order perturbation terms increases only exponentially, not like the combinatorial factors. Thus, we recognize that Efimov's result poses a difficulty more serious than considered by Baker.

There is an old standing hope that if the strong repulsion, in particular the hard sphere potential, could be treated appropriately, the attractive part of the interaction would be dealt with fairly easily. Thus, Lee, Huang and Yang developed the method of pseudopotential by generalizing the method due to Fermi.[8)9)] The method was not very successful because the pseudopotential admits the penetration of the wave function into the interacting spheres. Moreover, the potential is not Hermitian, and can not reproduce a complete set of eigenfunctions and eigenvalues.

In an earlier work, one of the present authors has proposed a new method for dealing with the hard sphere potential.[10)] It enables us to express the contributions to the three-particle correlation energy as convergent integrals which turn out to be of order $(k_0a)^4 \log k_0a$, where a is the radius of the hard sphere. A more precise estimation of the integrals opens the way to the calculation of the many non-precursor terms of order $(k_0a)^4$ which have been made obscure by the precursor terms. Here we discuss a typical precursor term in the three-particle correlation energy, i.e., the energy shift that the correlation causes via the off energy-shell self-energy of excited particles in a fermion system.

Three Particle Correlation and the Self Energy of Excited Particles

Contrary to the common perturbation scheme, we do not use the potential strength as the smallness parameter. Our smallness parameter is the hard sphere radius a. In the limit of long wave length $p^{-1} >> a$, the wave function ψ for the relative motion of two particles of mass m in free space can be expanded in powers of pa as

$$\psi = \frac{\sin p(r-a)}{pr} \approx \frac{\sin pr}{pr} - pa\frac{\cos pr}{pr} - \frac{p^2a^2}{2}\frac{\sin pr}{pr}\cdots.$$

The term of order pa on the right hand side of the above equation can be reproduced by the lowest order perturbation theory in terms of the effective potential

$$V_1 = \frac{\hbar^2}{ma}\delta(r-a).$$

The term of order $(pa)^2$ in ψ can not be reproduced by the second order perturbation formula solely in terms of V_1. We need additional potential V_2 which we treat as the second order effective potential. The wave function accurate to order p^2a^2 for $r > a$ as well as for $r < a$ is obtained when we take $V_2 = V_1$. This circumstance is not changed, provided $k_0a << 1$, when the pair of particles are immersed in the background of other fermions and prevented from scattering into the Fermi sea.

The matrix elements of V_1 and V_2 sandwiched between two plane wave states $\boldsymbol{q}, \boldsymbol{q}'$ and $\boldsymbol{p}, \boldsymbol{p}'$ are given by

$$(\boldsymbol{q}, \boldsymbol{q}'|V_1|\boldsymbol{p}, \boldsymbol{p}') = (\boldsymbol{q}, \boldsymbol{q}'|V_2|\boldsymbol{p}, \boldsymbol{p}') = \frac{4\pi a\hbar^2}{m\Omega}f(\boldsymbol{q}-\boldsymbol{q}')f(\boldsymbol{p}-\boldsymbol{p}')\delta^3(\boldsymbol{q}+\boldsymbol{q}'-\boldsymbol{p}-\boldsymbol{p}'),$$

where Ω is the volume of the system being considered, and

$$f(\boldsymbol{Q}) = f(Q) = \frac{\sin Qa/2}{Qa/2}. \qquad Q = |\boldsymbol{Q}|$$

The factor $f(\boldsymbol{Q})$ reflects the blocking effect of the strong repulsion on the penetration of particles inside the sphere. It enters in the integrand in the expression of energy as a form factor which prevents the divergence of the integral. In the limit $a \to 0$, $f(Q)$ tends to unity. But the approximation $f(\cdots) \approx 1$ is not justified within the integration symbol when the convergence of the integrals over the arguments of $f(\cdots)$ are poor. The nonuniformity of the convergence of energy integrals for small values of a is at the heart of difficulty which Efimov has encountered. It may cause divergence, if dealt with improperly.

We use V_1 and V_2 to study the effect of the three-particle correlation on the energy shift of hard sphere fermions due to the off energy-shell self-energy of excited particles. The interaction is assumed to be charge- and spin-independent. The multiplicity of a fermion is given by $\mu = (2S+1)(2T+1)$, where S and T are the spin and isospin.

The lowest order effect of the three-particle correlation results from processes in which an intermediate state involves three particles above the Fermi sea. Thus we need to take account of the interaction *in the intermediate* state to the second order in V_1 or to the lowest order in V_2. The energy shift of interest can be written as

$$E = w\frac{\Omega\hbar^2 a^3}{16m\pi^9}\int d^3\boldsymbol{p_1}\int d^3\boldsymbol{p_2}\int d^3\boldsymbol{p_3}\overline{\theta}_{\boldsymbol{p}_1}\overline{\theta}_{\boldsymbol{p}_2}\overline{\theta}_{\boldsymbol{p}_3}$$
$$\int d^3\boldsymbol{q}\frac{\theta_{\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{q}}\theta_{\boldsymbol{q}}f^2(\boldsymbol{p_1}-\boldsymbol{p_2})f^2(2\boldsymbol{q}-\boldsymbol{p_1}-\boldsymbol{p_2})f^2(\boldsymbol{q}-\boldsymbol{p_3})}{[(\boldsymbol{p_1}+\boldsymbol{p_2}-\boldsymbol{q})^2+q^2-p_1^2-p_2^2]^2}$$
$$\left\{1-\frac{a}{\pi^2}\int d^3\boldsymbol{r}\frac{\theta_{\boldsymbol{p}_3-\boldsymbol{r}}\theta_{\boldsymbol{q}+\boldsymbol{r}}f^2(2\boldsymbol{r}-\boldsymbol{p_3}+\boldsymbol{q})}{(\boldsymbol{q}+\boldsymbol{r})^2+(\boldsymbol{p_3}-\boldsymbol{r})^2+(\boldsymbol{p_1}+\boldsymbol{p_2}-\boldsymbol{q})^2-p_1^2-p_2^2-p_3^2}\right\}.$$

where $w = \mu(\mu-1)^2$ and

$$\theta_{\boldsymbol{q}} = \theta_q = 1-\overline{\theta}_q = \begin{cases} 1 & |\boldsymbol{q}| = q > k_0 \\ 0. & q < k_0 \end{cases}$$

The integral E is convergent. But ultraviolet divergence would result if we let $a \to 0$ in the integrand so that $f(\cdots) \to 1$. This indicates that large values of the wave numbers of excited particles, $\boldsymbol{q}$ and $\boldsymbol{r}$, are responsible for the main contribution to the integral E. Thus we can approximate E by

(1) dropping $\theta_{\boldsymbol{p}_3-\boldsymbol{r}}\theta_{\boldsymbol{q}+\boldsymbol{r}}$ in the expression in the curly bracket,

(2) neglecting the hole wave numbers $\boldsymbol{p_1}, \boldsymbol{p_2}$, and $\boldsymbol{p_3}$ in the energy denominators as compared to $\boldsymbol{q}$ and $\boldsymbol{r}$,

(3) neglecting $\boldsymbol{p_1}, \boldsymbol{p_2}$, and $\boldsymbol{p_3}$ in $f(\cdots)$ as well as in $\theta_{\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{q}}$ which restricts the range of $\boldsymbol{q}$ according to the Pauli principle.

The approximation E_0 to E so obtained is

$$E_0 = w\frac{\Omega\hbar^2 a^3}{16m\pi^9}\int d^3\boldsymbol{p_1}\overline{\theta}_{\boldsymbol{p}_1}\int d^3\boldsymbol{p_2}\overline{\theta}_{\boldsymbol{p}_2}\int d^3\boldsymbol{p_3}\overline{\theta}_{\boldsymbol{p}_3}\int d^3\boldsymbol{q}\theta_q\frac{f^2(q)}{2q^2}\frac{f^2(2q)}{2q^2}\left\{1-\frac{a}{\pi^2}\int d^3\boldsymbol{r}\frac{f^2(2\boldsymbol{r}+\boldsymbol{q})}{(\boldsymbol{q}+\boldsymbol{r})^2+r^2+q^2}\right\}.$$

Fortunately, E_0 can be reduced to a simple integral

$$E_0 = w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}\frac{16}{9\pi^3}k_0^4 a^3 \int_{k_0}^{\infty}\frac{dq}{q^2}\left(\frac{\sin aq}{aq}-\frac{\sin aq/2}{aq/2}\right)^2\left(1-\frac{1}{\sqrt{3}aq}(1-e^{-\sqrt{3}aq})\right)$$

thanks to

$$1-\frac{a}{\pi^2}\int d^3 r \frac{f^2(2r+q)}{(q+r)^2+r^2+q^2} = 1-\frac{1}{\sqrt{3}aq}(1-e^{-\sqrt{3}aq}).$$

The last expression for E_0 can be estimated accurately to order $(k_0a)^4$ by standard calculus.

$$\begin{aligned}E_0 &= w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}(k_0a)^4\left(\frac{2}{\sqrt{3}\pi}\right)^3\Bigg\{-\log k_0 a \\ &\quad -\gamma-\sqrt{3}\left(\frac{59}{360}\pi+\frac{1}{6}\arctan\frac{\sqrt{3}}{2}\right)+\frac{49}{20}-\frac{104}{135}\log 2-\frac{3}{4}\log 3+\frac{143}{2160}\log 7\Bigg\}+\cdots \\ &= w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}(k_0a)^4\left(-\left(\frac{2}{\sqrt{3}\pi}\right)^3\log k_0 a-0.022550\right),\end{aligned}$$

where $\gamma = 0.577215\cdots$ is Euler's constant.

In order to correct for the errors introduced by the above steps (1),(2) and (3) which have led us to E_0, we rewrite E as

$$E = E_0 + E_{1a} + E_{1b} + E_2,$$

and estimate approximately the terms E_{1a},E_{1b} and E_2 which are given by

$$\begin{aligned}E_{1a} &= w\frac{\Omega\hbar^2 a^3}{16m\pi^9}\int d^3 p_1\bar{\theta}_{p_1}\int d^3 p_2\bar{\theta}_{p_2}\int d^3 p_3\bar{\theta}_{p_3}\int d^3 q\theta_q\left[1-\frac{a}{\pi^2}\int d^3 r\frac{f^2(2r+q)}{(q+r)^2+r^2+q^2}\right] \\ &\quad \left\{\frac{\theta_{p_1+p_2-q}f^2(p_1-p_2)f^2(2q-p_1-p_2)f^2(q-p_3)}{[(p_1+p_2-q)^2+q^2-p_1^2-p_2^2]^2}-\frac{f^2(q)}{2q^2}\frac{f^2(2q)}{2q^2}\right\},\end{aligned}$$

$$\begin{aligned}E_{1b} &= w\frac{\Omega\hbar^2 a^4}{16m\pi^{11}}\int d^3 p_1\bar{\theta}_{p_1}\int d^3 p_2\bar{\theta}_{p_2}\int d^3 p_3\bar{\theta}_{p_3} \\ &\quad \int d^3 q\frac{\theta_q\theta_{p_1+p_2-q}f^2(p_1-p_2)f^2(2q-p_1-p_2)f^2(q-p_3)}{[(p_1+p_2-q)^2+q^2-p_1^2-p_2^2]^2} \\ &\quad \int d^3 r\left\{\frac{f^2(2r+q)}{(q+r)^2+r^2+q^2}-\frac{f^2(2r-p_3+q)}{(q+r)^2+(p_3-r)^2+(p_1+p_2-q)^2-p_1^2-p_2^2-p_3^2}\right\}\end{aligned}$$

and

$$\begin{aligned}E_2 &= w\frac{\Omega\hbar^2 a^4}{16m\pi^{11}}\int d^3 p_1\bar{\theta}_{p_1}\int d^3 p_2\bar{\theta}_{p_2}\int d^3 p_3\bar{\theta}_{p_3}\int d^3 q\,\theta_q\theta_{p_1+p_2-q}\frac{f^2(p_1-p_2)f^2(2q-p_1-p_2)f^2(q-p_3)}{[(p_1+p_2-q)^2+q^2-p_1^2-p_2^2]^2} \\ &\quad \int d^3 r\frac{(1-\theta_{p_3-r}\theta_{q+r})f^2(2r-p_3+q)}{(q+r)^2+(p_3-r)^2+(p_1+p_2-q)^2-p_1^2-p_2^2-p_3^2}.\end{aligned}$$

The expression in the square bracket in the equation for E_{1a} can be approximated by

$$1-\frac{a}{\pi^2}\int d^3\boldsymbol{r}\frac{f^2(2\boldsymbol{r}+\boldsymbol{q})}{(\boldsymbol{q}+\boldsymbol{r})^2+\boldsymbol{r}^2+\boldsymbol{q}^2} \quad = \quad 1-\frac{1}{\sqrt{3}aq}(1-e^{-\sqrt{3}aq}) \quad \approx \quad \frac{\sqrt{3}aq}{2},$$

while the estimation of E_{1b} is facilitated thanks to

$$\int d^3\boldsymbol{r}\left\{\frac{f^2(2\boldsymbol{r}+\boldsymbol{q})}{(\boldsymbol{q}+\boldsymbol{r})^2+\boldsymbol{r}^2+\boldsymbol{q}^2}-\frac{f^2(2\boldsymbol{r}-\boldsymbol{p_3}+\boldsymbol{q})}{(\boldsymbol{q}+\boldsymbol{r})^2+(\boldsymbol{p_3}-\boldsymbol{r})^2+(\boldsymbol{p_1}+\boldsymbol{p_2}-\boldsymbol{q})^2-p_1^2-p_2^2-p_3^2}\right\}$$

$$\approx\begin{cases}\pi^2(\sqrt{R(\boldsymbol{q},\boldsymbol{p_1},\boldsymbol{p_2},\boldsymbol{p_3})}-\frac{\sqrt{3}}{2}q), & R(\boldsymbol{q},\boldsymbol{p_1},\boldsymbol{p_2},\boldsymbol{p_3})>0\\ -\pi^2\frac{\sqrt{3}}{2}q, & R(\boldsymbol{q},\boldsymbol{p_1},\boldsymbol{p_2},\boldsymbol{p_3})<0\end{cases}$$

where

$$R(\boldsymbol{q},\boldsymbol{p_1},\boldsymbol{p_2},\boldsymbol{p_3})=\frac{1}{4}(q+p_3)^2+\frac{1}{2}[(\boldsymbol{p_1}+\boldsymbol{p_2}-\boldsymbol{q})^2-p_1^2-p_2^2-p_3^2].$$

The remaining factors $f(\cdots)$ in E_{1a} and E_{1b} can be approximated by unity. We put the two resultant expressions of E_{1a} and E_{1b} together to write

$$\begin{aligned}E_1 &= E_{1a}+E_{1b}\\ &= w\frac{\Omega\hbar^2a^4}{16m\pi^9}\int d^3\boldsymbol{p_1}\overline{\theta}_{p_1}\int d^3\boldsymbol{p_2}\overline{\theta}_{p_2}\int d^3\boldsymbol{p_3}\overline{\theta}_{p_3}\int d^3\boldsymbol{q}\theta_q\\ &\qquad\left\{\frac{\theta_{p_1+p_2-q}\sqrt{R(\boldsymbol{q},\boldsymbol{p_1},\boldsymbol{p_2},\boldsymbol{p_3})}}{[(\boldsymbol{p_1}+\boldsymbol{p_2}-\boldsymbol{q})^2+q^2-p_1^2-p_2^2]^2}-\frac{\sqrt{3}}{8}\frac{1}{q^3}\right\},\end{aligned}$$

where $\sqrt{R(\boldsymbol{q},\boldsymbol{p_1},\boldsymbol{p_2},\boldsymbol{p_3})}$ should be read as 0 if

$$R(\boldsymbol{q},\boldsymbol{p_1},\boldsymbol{p_2},\boldsymbol{p_3})<0.$$

For numerical estimation of the integral E_1, we find it convenient to use as variables of integration the absolute values of $\boldsymbol{t}=\boldsymbol{p_1}+\boldsymbol{p_2}$, $\boldsymbol{q_2}=\boldsymbol{t}-\boldsymbol{q}$, $\boldsymbol{u}=\boldsymbol{p_1}-\boldsymbol{t}/2$, and $\boldsymbol{s}=\boldsymbol{q}+\boldsymbol{p_3}$. Thus, putting

$$R=\frac{1}{4}s^2+\frac{1}{2}(q_2^2-2u^2-\frac{1}{2}t^2-p_3^2)$$

we obtain

$$\begin{aligned}E_1 &= w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2k_0^2}{2m}(k_0a)^4\frac{24}{\pi^3}\int_{k_0}^{\infty}dq\left\{-\frac{\sqrt{3}}{27q}+\frac{2}{k_0^9}\int_0^{2k_0}dt\int_{\max(k_0,|q-t|)}^{q+t}dq_2q_2\int_0^{\sqrt{k_0^2-t^2/4}}du u\right.\\ &\qquad\left.\frac{\min(tu,k_0^2-u^2-t^2/4)}{(q^2+q_2^2-2u^2-t^2/2)^2}\int_{\max(q-k_0,2q-\sqrt{2q^2+2q_2^2-4u^2-t^2})}^{q+k_0}ds\ s\int_{|q-s|}^{\min(k_0,\sqrt{q_2^2-2u^2+(s^2-t^2)/2})}dp_3\ p_3\sqrt{R}\right\}\\ &= w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2k_0^2}{2m}(k_0a)^4(-0.002293).\end{aligned}$$

In estimating the correction term E_2, we can put all form factors as unity. This is because the range of the vector $\boldsymbol{r}$ is bounded owing to the factor $(1-\theta_{\boldsymbol{p}_3-\boldsymbol{r}}\theta_{\boldsymbol{q}+\boldsymbol{r}})$ and that the energy denominator is of order $\max(q^6, q^4r^2)$ for large values of q and r. Thus

$$E_2 = w\frac{\Omega\hbar^2a^4}{16m\pi^{11}}\int d^3\boldsymbol{p}_1\overline{\theta}_{\boldsymbol{p}_1}\int d^3\boldsymbol{p}_2\overline{\theta}_{\boldsymbol{p}_2}\int d^3\boldsymbol{p}_3\overline{\theta}_{\boldsymbol{p}_3}\int d^3\boldsymbol{q}\,\theta_{\boldsymbol{q}}\theta_{\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{q}}\frac{1}{[(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{q})^2+q^2-p_1^2-p_2^2]^2}$$
$$\int d^3\boldsymbol{r}\frac{(1-\theta_{\boldsymbol{p}_3-\boldsymbol{r}}\theta_{\boldsymbol{q}+\boldsymbol{r}})}{(\boldsymbol{q}+\boldsymbol{r})^2+(\boldsymbol{p}_3-\boldsymbol{r})^2+(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{q})^2-p_1^2-p_2^2-p_3^2}\,.$$

Carrying out the integration over $\boldsymbol{r}$ we find

$$E_2 = w\frac{\Omega\hbar^2a^4}{16\pi^{10}m}\int d^3\boldsymbol{p}_1\overline{\theta}_{\boldsymbol{p}_1}\int d^3\boldsymbol{p}_2\overline{\theta}_{\boldsymbol{p}_2}\int d^3\boldsymbol{p}_3\overline{\theta}_{\boldsymbol{p}_3}$$
$$\int d^3\boldsymbol{q}\frac{\theta_{\boldsymbol{q}}\theta_{\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{q}}\Xi(\,|\boldsymbol{p}_3+\boldsymbol{q}|,\ R(\boldsymbol{q},\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)\,)}{[(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{q})^2+q^2-p_1^2-p_2^2]^2},$$

where $\Xi(|\boldsymbol{s}|,R)$ is defined by

$$\Xi(|\boldsymbol{s}|,R)=\frac{1}{2\pi}\int d^3\boldsymbol{q}\ \mathrm{H.W.}\ \frac{1-\theta_{\boldsymbol{q}+\boldsymbol{s}/2}\theta_{\boldsymbol{q}-\boldsymbol{s}/2}}{q^2+R},$$

in which the principal part integral is implied if $R<0$. The concrete expression of $\Xi(|\boldsymbol{s}|,R)$ can be given, in terms of the symbols $\kappa_0=k_0-s/2$ and $\kappa_2=k_0+s/2$, as follows

$$\Xi(s,R)=\begin{cases}\dfrac{\kappa_0\kappa_2+R}{s}\log\dfrac{\kappa_2^2+R}{\kappa_0\kappa_2+R}-2\sqrt{R}\arctan\dfrac{\kappa_2}{\sqrt{R}}+\kappa_2, & (R>0,\ k_0>s/2)\\ \dfrac{\kappa_0\kappa_2+R}{s}\log\dfrac{\kappa_2^2+R}{\kappa_0^2+R}-2\sqrt{R}\arctan\dfrac{2k_0\sqrt{R}}{R-\kappa_0\kappa_2}+2k_0, & (R>0,\ k_0<s/2)\\ \dfrac{\kappa_0\kappa_2+R}{s}\log\left|\dfrac{\kappa_2^2+R}{\kappa_0\kappa_2+R}\right|+\sqrt{-R}\log\left|\dfrac{\kappa_2-\sqrt{-R}}{\kappa_2+\sqrt{-R}}\right|+\kappa_2, & (R<0,\ k_0>s/2)\\ \text{We need not consider the case} & (R<0,\ k_0<s/2).\end{cases}$$

In this way we can evaluate E_2 as

$$E_2=w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2k_0^2}{2m}(k_0a)^4\frac{48}{\pi^4k_0^9}\int_{k_0}^{\infty}dq\int_0^{2k_0}dt\int_{\max(k_0,|q-t|)}^{q+t}dq_2q_2\int_0^{\sqrt{k_0^2-t^2/4}}du$$
$$\frac{u\min(tu,k_0^2-u^2-t^2/4)}{(q^2+q_2^2-2u^2-t^2/2)^2}\int_{q-k_0}^{q+k_0}dss\int_{|q-s|}^{k_0}dp_3p_3\,\Xi\Big(s,\ \tfrac{1}{4}s^2+\tfrac{1}{2}(q_2^2-2u^2-\tfrac{1}{2}t^2-p_3^2)\Big)$$
$$=w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2k_0^2}{2m}(k_0a)^4(0.012257).$$

Conclusion

We find the three-particle correlation energy due to the off energy-shell self-energy of excited particles in a fermion system of hard spheres as

$$E=-w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2k_0^2}{2m}(k_0a)^4(0.0496545\log k_0a+0.012586).$$

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(The symbols $E, E_1, E_{1a}, E_{1b}, E_2$ in the following **Errata** stand for quantities different from the ones implied by the same symbols in the above text)

Errata

Renormalized Ring Diagram Energy in a Fermion System of Hard Spheres
S.Yamasaki and T.Yamanishi
Memoirs of Fukui University of Technology, 32 Part 1, 25,(2004).

The last equation on p31 should read

$$E_1 = w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}(k_0 a)^4(-0.008910).$$

The next to last equation on p32 should read

$$E_2 = w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}(k_0 a)^4(0.007914).$$

The last equation on p32 should read

$$E = w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}(k_0 a)^4(-0.0496545\log k_0 a - 0.025547).$$

（平成16年11月26日受理）