

# On a Type of Modules over a Ring of Global Dimension Three

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## 1. Introduction

In the modules over a ring of global dimension 3 a type of modules, called 1-simple modules, has notable properties. In this paper we show some of these properties.

Throughout this note,  $R$  denotes an artinian ring with an identity element,  $J$  denotes the Jacobson radical of  $R$ , and all modules are assumed to be unitary. The projective dimension of a module  $M$  is denoted by  $\text{pd}(M)$ . The symbol of inequality  $<$  (resp.  $>$ ) is in the strict sense, in other words, it represents  $\leq$  (resp.  $\geq$ ) and  $\neq$ .

## 2. 1-simple modules and semi-1-simple modules

In this paper, a module  $S$  is called 1-simple if it satisfies the following conditions :

- (1)  $\text{pd}(S) \leq 1$ ,
- (2)  $\text{pd}(X) \geq 2$  for any submodule  $0 < X < S$ .

A finite direct sum of 1-simple modules is called a semi-1-simple module, and a finite direct sum of copies of one 1-simple module  $S$  is called  $(S)$ -homogeneous semi-1-simple module. In the category of modules over a ring of global dimension 3, these types of modules play special rolls analogous to those of simple modules, semisimple modules and homogeneous semi-simple modules in the general situation.

Lemma 1. Let  $0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$  be a short exact sequence of  $R$ -modules, and let  $k = \text{pd}(K)$ ,  $m = \text{pd}(N)$ . If  $m < \max(k, n)$ , then we have  $n = k + 1$ .

Proof. For an arbitrary  $R$ -module  $X$ , the following long exact sequence is derived from the given short exact sequence :

$$\begin{aligned} \cdots \rightarrow \text{Ext}^{m+1}(N, X) \rightarrow \text{Ext}^{m+1}(M, X) \rightarrow \text{Ext}^{m+1}(K, X) \rightarrow \text{Ext}^{m+2}(N, X) \\ \rightarrow \text{Ext}^{m+2}(M, X) \rightarrow \text{Ext}^{m+2}(K, X) \rightarrow \text{Ext}^{m+3}(N, X) \rightarrow \cdots \end{aligned}$$

In this long exact sequence  $\text{Ext}^i(M, X) = 0$  ( $i = m + 1, m + 2, m + 3, \cdots$ ) and the homomorphisms  $\text{Ext}^i(K, X) \rightarrow \text{Ext}^{i+1}(N, X)$  are isomorphisms. These isomorphisms

show that  $n=k+1$  whenever  $m < \max(k, n)$ .

Lemma 2. Let  $\text{gl dim } R = 3$ . If  $K$  is a submodule of a module  $M$  and  $\text{pd}(M) \leq 2$ , then  $\text{pd}(K) \leq 2$ .

Proof. If  $K \leq M$ ,  $\text{pd}(M) \leq 2$  and  $\text{pd}(K) = 3$ , then  $\text{pd}(M/K) = 4$  by Lemma 1, which contradicts to  $\text{gl dim } R = 3$ .

The following two propositions show the property of 1-simple modules and that of semi-1-simple modules. These properties are important especially in the case of  $\text{gl dim } R = 3$ .

Proposition 3. Let  $\text{gl dim } R = 3$ . Then we have :

- (1) Every homomorphism from a 1-simple module to a module of projective dimension  $\leq 2$  is an isomorphism or zero,
- (2) If  $K \leq M$ ,  $\text{pd}(M) \leq 2$ ,  $\text{pd}(K) \leq 1$  and  $S$  is a 1-simple submodule of  $M$ , then  $S \leq K$  or  $S \cap K = 0$ .

Proof. (1). Let  $X$  be the kernel of the homomorphism. Suppose  $0 < X < S$ , then the image of  $S$  is of projective dimension 3 by Lemma 1, which is impossible by Lemma 2.

(2). By Lemma 2  $\text{pd}(S+K) \leq 2$ . And  $\text{pd}(S+K/S) \leq 2$  by Lemma 1. The isomorphism  $S+K/S \cong K/S \cap K$  tells us that  $\text{pd}(K/S \cap K) \leq 2$ . If  $0 < S \cap K < S$ , then  $\text{pd}(S \cap K) = 2$  by the 1-simplicity of  $S$ , and  $\text{pd}(K/S \cap K) = 3$ , which is a contradiction. Therefore  $0 = S \cap K$  or  $S \cap K = S$ .

Proposition 4. Every homomorphic image of a semi-1-simple module in a module of projective dimension  $\leq 2$  is semi-1-simple.

Proof. Let  $S_i$  be 1-simple modules ( $i = 1, \dots, n$ ) and  $T = \bigoplus_{i=1}^n S_i$ . We prove by induction on  $n$ . If  $n = 1$ , the assertion is identical to Proposition 3, (1). The image of  $T$  is the sum of the image of  $\bigoplus_{i=1}^{n-1} S_i$  and the image of  $S_n$ . Both images are semi-1-simple by the hypothesis of induction. On the other hand, the intersection of the image is 0 or the image of  $S_n$  is contained in the image of  $\bigoplus_{i=1}^{n-1} S_i$  by Proposition 3, (2). Therefore the image of  $\bigoplus_{i=1}^n S_i$  is semi-1-simple.

Proposition 5. Let  $M = \bigoplus_{i=1}^n S_i$  be a semi-1-simple module with 1-simple modules  $S_i$ ,

and let  $N$  be a submodule of  $M$  whose projective dimension is 1. Then  $N$  is a direct summand of  $M$ , and  $N$  is semi-1-simple.

Proof. Let  $n$  be the number of the indexes  $k$  such that  $S_k$  is not included in  $N$ . We show that  $M = N \oplus (\bigoplus_{i \in I} S_i)$  for some subset  $I$  of  $\{1, \dots, n\}$  by induction on  $n$ . If  $n = 0$ , the assertion is evident. If  $n > 0$ , there exists an index  $1 \leq k \leq n$  such that  $S_k$  is not included in  $N$ , and then  $S_k \cap N = 0$  by Proposition 3 (2). By the hypothesis of induction  $M = (N \oplus S_k) \oplus (\bigoplus_{i \in J} S_i)$  for some subset  $J$  of  $\{1, \dots, n\}$ . Therefore  $M = N \oplus (\bigoplus_{i \in I} S_i)$  for some subset  $I$  of  $\{1, \dots, n\}$ , from which it follows immediately that  $N$  is isomorphic to  $\bigoplus_{i \notin I} S_i$ .

### 3. Traces of a 1-simple module

Let  $M$  and  $N$  be modules. The sub-module  $\sum_{f \in \text{Hom}(N, M)} f(N)$  in  $M$  is called the trace of  $N$  in  $M$ . This is usually denoted by  $\text{trace}_M(N)$ . However the trace of  $N$  in  $M$  is denoted by  $T(M)$  in this paper:  $T_N(M) = \text{trace}_M(N)$ .

Lemma 6. Let  $\text{gl dim } R = 3$ ,  $S$  be a 1-simple module, and let  $T = T_s(R)$ .

- (1) If  $M$  is of projective dimension  $\leq 2$ , then  $T_s(M)$  is semi-1-simple  $S$ -homogeneous.
- (2) If  $M_1$  and  $M_2$  are modules of projective dimension  $\leq 1$ , then
 
$$T_s(M_1 \oplus M_2) = T_s(M_1) \oplus T_s(M_2).$$
- (3) If  $P$  is a projective module, then  $T_s(P) = PT$ .

Proof. (1). Any homomorphic image of  $S$  is isomorphic to  $S$  or 0 by Proposition 3 (1), and the sum of all homomorphic images of  $S$  is a direct sum of these images by Proposition 3 (2).

(2) follows from (2) and Proposition 3 (2) immediately.

(3) For a direct sum of copies of  $R$ , (3) is evident. As  $P$  is projective, there exists such a module  $P'$  that  $P \oplus P'$  is a direct sum of copies of  $R$ . Therefore  $T_s(P) \oplus T_s(P') = T_s(P \oplus P') = (P \oplus P') T = PT \oplus P' T$  by (2). The intersection of both terminals of this equation with  $P$  are  $T_s(P)$  and  $PT$ , which are equal.

Lemma 7. Let  $\text{gl dim } R = 3$ ,  $J$  be the Jacobson radical of  $R$ ,  $S$  be a 1-simple module, and let  $T = T_s(R)$ . Then  $JT = 0$ .

Proof. The multiplication of an element of  $J$  can not be an isomorphism of a non-

zero right ideal. Since  $T$  is a sum of 1-simple right ideals,  $JT=0$  by Proposition 3 (1).

Proposition 8. Let  $\text{gl dim } R=3$ ,  $S$  be 1-simple and not projective,  $T=T_s(R)$ , and

$$0 \longrightarrow K \xrightarrow{k} P \xrightarrow{p} S \longrightarrow 0$$

be an exact sequence with projective cover  $p$  of  $S$  and its kernel  $k$ . Then  $PT=KT=0$ .

Proof. First we remark that  $K$  is projective. If  $T(K) < T(P)$ , then  $P$  is a direct sum of  $K$  and a 1-simple submodule in  $P$  not in  $K$ , by the equality of the length of  $P$  and the length of this direct sum, which contradicts to the non-projectivity of  $S$ . Therefore  $PT=T_s(P)=T_s(K)=KT$ . Since  $p$  is the projective cover of  $S$ , we have  $PJ \geq K$  and  $0=PJT \geq KT$ . Hence  $PT=KT=0$ .

#### 4. Self-extension of 1-simple modules

Let  $S$  be a simple module of projective dimension 1. Then every extension of  $S$  by  $S$  is semi-simple or uniserial. Suppose that a uniserial extension  $U$  of  $S$  by  $S$  exists. Then the Jacobson radical of the projective cover of  $U$  is projective and contains a submodule isomorphic to the projective cover of  $S$  as a direct summand, which contradicts to that the projective covers of  $S$  and  $U$  are isomorphic. Therefore any extension of  $S$  by  $S$  is semi-simple. This is generalized to the following theorem.

Theorem 9. Let  $\text{gl dim } R=3$  and  $S$  be a 1-simple module. Then  $\text{Ext}(S,S)=0$ .

$$a \quad b$$

Proof. Let  $0 \longrightarrow S_1 \xrightarrow{a} X \xrightarrow{b} S_2 \longrightarrow 0$  ( $S_1 \cong S_2 \cong S$ ) be an exact sequence. Then  $\text{pd}(X)=1$ , and by Proposition 3 (2) there are only two cases. One is the case that there exists a submodule  $Y \leq X$  with  $\text{pd}(Y) \leq 1$  such that  $Y \cap a(S_1)=0$ . The other is the case that  $Y \geq a(S_1)$  for any submodule  $0 \neq Y \leq X$  with  $\text{pd}(Y) \leq 1$ . In the first case,  $Y$  is isomorphic to a submodule of  $S_2$ , and  $Y \cong S_2$  by the 1-simplicity of  $S_2$ , and  $X = a(S_1) \oplus Y$  by the equality of lengths. Then the exact sequence splits. In the second case,  $X/Y$  is a factor module of  $S_2$  and  $\text{pd}(X/Y) \leq 2$ . Then  $Y = a(S_1)$  or  $Y = X$  by Proposition 3 (1). In other words  $0, a(S_1), X$  are the only submodules of projective dimension  $\leq 1$  in  $X$ , in the second case.

We show that the second case is impossible. Suppose the second case, and let  $c$  be an isomorphism from  $S_2$  to  $S_1$ . Then  $r=acb$  is an endomorphism of  $X$ , and  $r^2=0$ . Let

$p : P \rightarrow X$  be the projective cover of  $X$ . Then, by the projectivity of  $P$ , there exists an endomorphism  $s$  of  $P$  which makes the following diagram commutative :

$$\begin{array}{ccc} & p & \\ & P \longrightarrow X & \\ s \downarrow & & \downarrow r \\ & P \longrightarrow X & \\ & p & \end{array}$$

Then the image  $s^2(P)$  is contained in the Jacobson radical of  $P$ , and  $r^{-1}(0) = a(S_1)$ . Since there is an epimorphism from  $X$  to  $S_2$ , the projective cover of  $S_2$  is isomorphic to a direct summand  $Q$  of  $P$ . By Proposition 8 we have,  $T_s(Q) = 0$ . Therefore  $s(Q)$  is isomorphic to  $Q$ ,  $s^2(Q)$  is isomorphic to  $s(Q)$ , and  $s^2(Q) \leq PJ$ . On the other hand  $P$  is isomorphic to a direct summand of the direct sum  $P(S_1) \oplus P(S_2)$  of the projective covers  $P(S_1) (\cong P(S_2))$  and  $P(S_2)$  of  $S_1$  and  $S_2$ , since  $X$  is an extension of  $S_1$  by  $S_2$ . Hence  $P(S_1) \oplus P(S_2)$  has a submodule isomorphic to  $P(S_2)$  in the Jacobson radical  $(P(S_1) \oplus P(S_2))J$ , which contradicts to the equality of the Loewy length of  $P(S_1) \cong P(S_2)$  and the Loewy length of the submodule.

## REFERENCES

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