On the Solution of Some Functional Equations by Matrix Calculus

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In the previous papers, mixed boundary value problems in a complicated domain were reduced to the solution of functional equations expressing the given boundary conditions and, if necessary, the conditions for the merging of functions separately defined in partial domains into a smoothly continuous one. The solution was given by a series of repeated contour integrals in some simpler cases. In this paper it is shown that the numerical computation may be facilitated by transforming the functional equations into matrix equations by use of integral transforms proposed earlier and an example is given.

1. Introduction

In treating the problems of mixed boundary value problems in a complicated domain, it may be found convenient to divide the whole domain into some pieces of partial ones, in which unknown functions satisfying the given partial differential equation are separately defined and afterwards they are made to merge into one smoothly continued solution. In the previous papers[1)], the method of expressing functions by contour integrals was found very useful to establish the functional equations to satisfy the conditions for the merging of functions into a smoothly continuous one at the boundaries of the partial domains. It was also shown[2)] that in simpler cases the solution of such functional equations may be given by a series of repeated contour integrals, but the method was not convenient to actual numerical computation.

To facilitate the computation, it would be more convenient to use a series of matrices instead of that of repeated integrals. For this purpose, some method of conversion of the contour integrals into matrix forms would be necessary.

In the following, it will be shown that the conversion may be fulfilled by expressing arbitrary functions in integral transforms, the residues at the poles of the integrand being considered to form vectors.

The procedure in detail, however, varies from one problem to another, so that only an example will be shown in the following to explain the principle involved.

2. Expression of Arbitrary Functions by Contour Integrals [1)]

Consider the ordinary differential equation of the second order in the interval (x_1, x_2) in the form

$$\frac{1}{P(x)}\frac{d}{dx}\left\{Q(x)\frac{du}{dx}\right\}+\left\{\lambda+R(x)\right\}u=0, \tag{1}$$

where $P(x)$, $Q(x)$ and $R(x)$ are given real functions, $P(x)$ and $Q(x)$ not vanishing in the interval and λ is a complex parameter.

It is known that the equation has a solution, say $S_1(x, \lambda)$, such that the values $S_1(x_1, \lambda)$ and $S_1'(x_1, \lambda)$ at the left end of the interval x_1 are independent of λ. A prime denotes the derivative with regard to x. There exists another solution $S_2(x, \lambda)$, such that the values $S_2(x_2, \lambda)$ and $S_2'(x_2, \lambda)$ at the right end of the interval x_2 are independent of λ. When the length of the interval is finite, these are integral functions of λ.

For any value of x in the interval, the expression

$$\{S_1'(x, \lambda)\,S_2(x, \lambda)-S_1(x, \lambda)\,S_2'(x, \lambda)\}\,Q(x)=E(\lambda) \tag{2}$$

is independent of x. If $E(\lambda)\neq 0$, $S_1(x, \lambda)$ and $S_2(x, \lambda)$ are two independent solutions of the equation (1).

The function $E(\lambda)$ has infinite number of simple zeros all lying on the real axis of λ[1)]. There exists a smallest one, say λ_1, hence they may be denoted by λ_n so as to be $\lambda_1<\lambda_2<\lambda_3<\cdots\cdots$. If $\lambda=\lambda_n$, $S_1(x, \lambda_n)$ and $S_2(x, \lambda_n)$ are no longer independent, but the ratio $S_1(x, \lambda_n)/S_2(x, \lambda_n)$ is a constant depending on λ_n.

For an arbitrary real function $F(x)$, let

$$\left.\begin{aligned}\psi_1(\lambda)&=\int_{x_1}^{x_2}F(x)S_1(x, \lambda)P(x)\,dx,\\ \psi_2(\lambda)&=\int_{x_1}^{x_2}F(x)S_2(x, \lambda)P(x)\,dx,\end{aligned}\right\} \tag{3}$$

both supposed to exist. The functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$ will be called S_1- and S_2- transforms of $F(x)$, respectively. Then in the interval where $F(x)$ is continuous,

$$\left.\begin{aligned}F(x)&=\frac{1}{2\pi i}\int_R\frac{S_1(x, \lambda)}{E(\lambda)}\psi_2(\lambda)\,d\lambda,\\ F(x)&=\frac{1}{2\pi i}\int_R\frac{S_2(x, \lambda)}{E(\lambda)}\psi_1(\lambda)\,d\lambda,\end{aligned}\right\} \tag{4}$$

hold[1], in which the paths of integrations R are taken to start from infinity under the real axis running along the lower side of all the poles of the integrand and after encircling the smallest pole at λ_1 clockwise, extending to infinity along the upper side of the real axis as shown in Fig. 1.

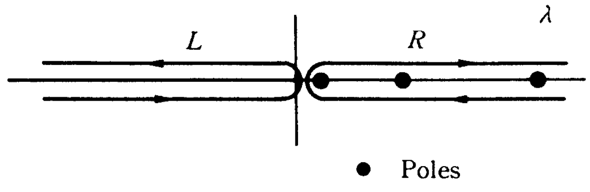

Fig. 1. Paths of integration.

The integrals (4), when expanded into series by taking the residues at the poles of the integrands, are reduced to the same form, which is no other than the Stone-Wyle-Titchmarsh-Kodaira's expansion.

It must be noted, however, that the functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are determined by the boundary condition at one end, x_1 and x_2 respectively, while the coefficients in the series depend on those at the both ends of the interval. That is to say, the value of $F(x)$ is incorporated in $\psi_1(\lambda)$ and $\psi_2(\lambda)$ together with the boundary condition at the respective end. The function $F(x)$ is expressible in series-form only when the boundary condition at the remaining end of the interval is specified by giving $S_2(x, \lambda)$ or $S_1(x, \lambda)$.

Especially interesting is the case when the interval is divided into two partial intervals at a point x_c, $(x_1 < x_c < x_2)$, and $F(x)$ is given in one of the partial intervals only. For instance, if $F(x)$ is given in the interval $x_1 < x < x_c$, then

$$\psi_1(\lambda) = \int_{x_1}^{x_c} F(x) S_1(x, \lambda) P(x) dx. \tag{5}$$

When x lies in the interval II, where $x_c < x < x_2$, the path of integration of the second integral (4) may be swung round to the left as shown by L in Fig. 1, because the integrand diminishes rapidly as $|\lambda| \to \infty$. Since the integrand is analytic on the left half plane of λ , the integral vanishes, so that

$$F(x) = 0, \quad (x_c < x < x_2) \tag{6}$$

is secured.

It is also to be noted that the function $\psi_1(\lambda)$ is not influenced by x_2. Hence the end point x_2 and the function $S_2(x, \lambda)$ may be changed insofar as $x_2 < x_c$ is secured.

These two facts play important roles for the transformation of the functional equation into matrix form.

Similar arguments hold also when $F(x)$ is given in the interval II, the suffices 1 and 2 being interchanged.

In the above, the functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are considered to be integral functions defined by equation (3). It is not necessary to restrict these functions as such in the integral (4), but then the relation (3) is no longer valid.

3. The Problem of Two Partial Intervals as a Simple Example

The problem is to find a function satisfying a given partial differential equation in a rectangular domain subject to the boundary conditions differeing in two partial intervals on one side of the domain but uniformly given on other sides.

The function $u(x, y)$ satisfying the partial differential equation is assumed to be expressed by a contour integral of the type (4) containing two variables x and y. Then the boundary value $U(x)$ and the gradient $W(x)$ on the side under consideration will be given by

$$\left.\begin{aligned} U(x) &= \frac{1}{2\pi i}\int_R \frac{S_i(x,\lambda)}{E(\lambda)}\psi_j(\lambda)\,d\lambda \\ \text{and}\qquad W(x) &= \frac{1}{2\pi i}\int_R \frac{S_i(x,\lambda)}{E(\lambda)}\varphi_j(\lambda)\,d\lambda, \end{aligned}\right\}\left(\begin{matrix} i, j = 1, 2 \\ i \neq j \end{matrix}\right), \tag{7}$$

where $\psi_j(\lambda)$ and $\psi_j(\lambda)$ are connected by

$$\left.\begin{aligned} \psi_j(\lambda) &= R(\lambda)\,\varphi_j(\lambda) \\ \text{and}\qquad k(\lambda) &= D_1(\lambda)\,/\,D_2(\lambda) \end{aligned}\right\}(j=1,2), \tag{8}$$

in which $D_1(\lambda)$ and $D_2(\lambda)$ are given function of λ, analytic on the right half plane of λ. No zeros of $D_1(\lambda)$ and $D_2(\lambda)$ are assumed to be coincident with any zeros of $E(\lambda)$. Hence the paths of integrations of equations (7) may be taken so as to have all the zeros of $E(\lambda)$ on their right and those of $k(\lambda)$ on their left.

When $W(x)$ is given and $\varphi_j(\lambda)$, $(j=1,2)$, are defined by the equation (3), $U(x)$ may be given by

$$\left.\begin{aligned} U(x) &= \frac{1}{2\pi i}\int_R \frac{S_i(x,\lambda)}{E(\lambda)}\psi_j(\lambda)\,d\lambda \\ &= \frac{1}{2\pi i}\int_R \frac{S_i(x,\lambda)}{E(\lambda)}k(\lambda)\,\varphi_j(\lambda)\,d\lambda \\ &= \frac{1}{2\pi i}\int_R \frac{S_i(x,\lambda)}{E(\lambda)}k(\lambda)\,d\lambda\left\{\int_{x_1}^{x_2} W(\xi)\,S_j(\xi,\lambda)\,P(\xi)\,d\xi.\right. \end{aligned}\right\} \tag{9}$$

If the change of the order of the integration is permissible, the relation is expressed as an integral equation.

$$\begin{aligned} U(x) &= \int_{x_1}^{x_2} W(\xi)\, P(\xi)\, d\xi \left\{ \frac{1}{2\pi i} \int_R \frac{S_i(x,\lambda)\, S_j(\xi,\lambda)}{E(\lambda)} k(\lambda)\, d\lambda \right\} \\ &= \int_{x_1}^{x_2} K(x,\xi)\, W(\xi)\, d\xi \qquad (i \neq j), \end{aligned} \tag{10}$$

where

$$\left.\begin{aligned} K(x,\xi) &= \frac{P(\xi)}{2\pi i} \int_R \frac{S_1(x,\lambda)\, S_2(\xi,\lambda)}{E(\lambda)} k(\lambda)\, d\lambda \\ &= \frac{P(\xi)}{2\pi i} \int_R \frac{S_2(x,\lambda)\, S_1(\xi,\lambda)}{E(\lambda)} k(\lambda)\, d\lambda. \end{aligned}\right\} \tag{11}$$

The equality of both forms of (11) is secured by the fact that for any value of λ_n satisfying $E(\lambda_n)=0$,

$$\frac{S_1(x,\lambda_n)}{S_2(x,\lambda_n)} = \frac{S_1(\xi,\lambda_n)}{S_2(\xi,\lambda_n)}. \tag{12}$$

When the integrands of (11) $=O(\lambda^{-\varepsilon})$, $(0<\varepsilon<1)$, the integrals for $K(\lambda,\xi)$ converge for $x \neq \xi$, but $K(x,\xi)\rightarrow\infty$ as $(x-\xi)\rightarrow 0$. Hence, the integral equation is of the first kind with a singular kernel.

The integral equation (10) may be solved very simply when $U(x)$ is prescribed for the whole interval (x_1, x_2). Namely, $\psi_j(\lambda)$ may be defined by the equation (3), so that $W(x)$ may be given by

$$W(x) = \frac{1}{2\pi i} \int_R \frac{S_i(x,\lambda)}{E(\lambda)} \frac{\psi_j(\lambda)}{k(\lambda)}\, d\lambda, \qquad (i, j = 1, 2,\ i = j). \tag{13}$$

The problem of partial intervals is stated as follows:
Suppose that the interval (x_1, x_2) is divided at the point $x_c\,(x_1 < x_c < x_2)$ into two partial intervals I: $(x_1 < x < x_c)$ and II: $(x_c < x < x_2)$. The function $U(x)$ is prescribed for the interval I, while $W(x)=0$ in the interval II. Find $W(x)$ for the interval I and $U(x)$ for the interval II.

4. Expression in Matrix Equation and its Solution

Numerical calculation of the solution of a problem of partial intervals may be facilitated when the equation is transformed into a matrix form.

Let $U^{(1)}(x)$ and $W^{(1)}(x)$ be the values in the interval I and $U^{(2)}(x)$ and $W^{(2)}(x)$ be those in the interval II. They are supposed to be expressed by

$$\left.\begin{aligned} U^{(1)}(x) &= \sum_{n=1}^{\infty} \psi_n^{(1)} S_1(x, \beta_n), \\ W^{(1)}(x) &= \sum_{n=1}^{\infty} \varphi_n^{(1)} S_1(x, \gamma_n), \end{aligned}\right\} \tag{14}$$

and

$$\left.\begin{aligned} U^{(2)}(x) &= \sum_{n=1}^{\infty} \psi_n^{(2)} S_2(x, \beta_n), \\ W^{(2)}(x) &= \sum_{n=1}^{\infty} \varphi_n^{(2)} S_2(x, \gamma_n), \end{aligned}\right\} \tag{15}$$

where β_n's and γ_n's are sets of constants suitably chosen.

The coefficients $\psi_n^{(1)}$'s, $\psi_n^{(2)}$'s, $\varphi_n^{(1)}$'s and $\varphi_n^{(2)}$'s may be considered that they are elements of the vectors $\psi^{(1)}$, $\psi^{(2)}$, $\varphi^{(1)}$ and $\varphi^{(2)}$ respectively, being connected by the relations

$$\left.\begin{aligned} \psi^{(1)} &= A_{11}\varphi^{(1)} + A_{12}\varphi^{(2)}, \\ \psi^{(2)} &= A_{21}\varphi^{(1)} + A_{22}\varphi^{(2)}, \end{aligned}\right\} \tag{16}$$

and

$$\left.\begin{aligned} \varphi^{(1)} &= B_{11}\psi^{(1)} + B_{12}\psi^{(2)}, \\ \varphi^{(2)} &= B_{21}\psi^{(1)} + B_{22}\psi^{(2)}, \end{aligned}\right\} \tag{17}$$

where A_{ij}'s and B_{ij}'s are matrices of coefficients.

Substituting (16) into the right-hand side of (17), and comparing both sides,

$$\left.\begin{aligned} A_{11}B_{11} + A_{12}B_{21} &= E, \quad A_{11}B_{12} + A_{12}B_{22} = 0, \\ A_{21}B_{11} + A_{22}B_{21} &= 0, \quad A_{21}B_{12} + A_{22}B_{22} = E \end{aligned}\right\} \tag{18}$$

are obtained, where E denotes unit matrix. Similarly, by substituting (17) into the right-hand side of (16) and comparing both sides,

$$\left.\begin{aligned} B_{11}A_{11} + B_{12}A_{21} &= E, \quad B_{11}A_{12} + B_{12}A_{22} = 0, \\ B_{21}A_{11} + B_{22}A_{21} &= 0, \quad B_{21}A_{12} + B_{22}A_{22} = E \end{aligned}\right\} \tag{19}$$

are obtained. Since $\varphi^{(2)} = 0$, the problem is reduced to that of finding $\varphi^{(1)}$ satisfying

$$A_{11}\varphi^{(1)} = \psi^{(1)}, \tag{20}$$

which is no other than the problem of the inversion of a matrix.

Multiplying bothe sides of the equation (20) by B_{11} on the left, the relation may be transformed into

$$B_{11}\psi^{(1)} = B_{11}A_{11}\varphi^{(1)} = (E-B_{12}A_{21})\varphi^{(1)}. \tag{21}$$

The inverse matrix $(E-B_{12}A_{21})^{-1}$ may be given by Neumann or Fredholm series. Then the solution may be written as

$$\left.\begin{aligned} &\varphi^{(1)} = (E-B_{12}A_{21})^{-1}B_{11}\psi^{(1)} \\ \text{and}\quad &\psi^{(2)} = B_{21}\varphi^{(1)} = B_{21}(E-B_{12}A_{21})^{-1}B_{11}\psi^{(1)}. \end{aligned}\right\} \tag{22}$$

When expressed by Neumann series[3], they may also be given by

$$\begin{aligned} \varphi^{(1)} &= \{E+B_{12}A_{21}+(B_{12}A_{21})^2+\cdots\cdots\}B_{11}\psi^{(1)} \\ &= B_{11}\{E+A_{12}B_{21}+(A_{12}B_{21})^2+\cdots\cdots\}\psi^{(1)} \\ &= B_{11}(E-A_{12}B_{21})^{-1}\psi^{(1)}, \end{aligned} \tag{23a}$$

and

$$\begin{aligned} \psi^{(2)} &= A_{21}B_{11}(E-A_{12}B_{21})^{-1}\psi^{(1)} \\ &= -A_{22}B_{21}(E-A_{12}B_{21})^{-1}\psi^{(1)} \\ &= -A_{22}(E-B_{21}A_{12})^{-1}B_{21}\psi^{(1)} \end{aligned} \tag{23b}$$

5. Expression of Matrices A_{12} and B_{21}

When the function $U^{(1)}(x)$ is prescribed in the interval I, where $x_1 < x_c < x_2$, the coefficients $\psi_n^{(1)}$'s in the series

$$U^{(1)}(x) = \sum_{n=1}^{\infty}\psi^{(1)} S_1(x, \beta_n) \tag{24}$$

may be determined as the residues at the poles of the integrand in (7), β_n 's being the zeros of $E(\lambda)$. But in general, β_n 's are not restricted as such. For instance, they may be taken to be zeros of $\lambda p D_2(\lambda)$, where p is 0 or an integer. For simplicity, p will be assumed to be 0 or 1. In the latter case, $\lambda=\beta_0=0$ is a simple pole in the integrand. If $p \geqq 2$, some modification may be necessary.

The S_1-transform of $U^{(1)}(x)$ is then expressed as

$$\begin{aligned} \psi_1(\lambda) &= \int_{x_1}^{x_c} U^{(1)}(x)\, S_1(x,\lambda)\, P(x)\, dx \\ &= \sum_{n=1}^{\infty}\psi_n^{(1)}\frac{[S_1'(x,\beta_n)S_1(x,\lambda)-S_1(x,\beta_n)S_1'(x,\lambda)]_{x_1}^{x_c}}{\lambda-\beta_n} \\ &= \sum_{n=1}^{\infty}\psi_n^{(1)}\frac{S_1'(x_c,\beta_n)S_1(x_c,\lambda)-S_1(x_c,\beta_n)S_1'(x_c,\lambda)}{\lambda-\beta_n} \end{aligned} \tag{25}$$

so that $W(x)$ corresponding to $U^{(1)}(x)$ in the whole interval is given by

$$W(x)=\frac{1}{2\pi i}\int\frac{S_2(x,\lambda)\,dx}{E(\lambda)\,k(\lambda)}\sum_{n=1}^{\infty}\psi_n^{(1)}\left\{\frac{S_1'(x_c,\beta_n)S_1(x_c,\lambda)-S_1(x_c,\beta_n)S_1'(x_c,\lambda)}{\lambda-\beta_n}\right\} \tag{26}$$

When x lies in the interval II $(x_c < x < x_2)$, the path of integration may be taken on R or L in Fig. 1. If the path L is taken, γ_m's are zeros of $D_1(\lambda)$. In this case, the coefficients $\varphi_m^{(1)}$'s in the series

$$W^{(2)}(x)=\sum_{n=1}^{\infty}\varphi_m^{(2)}S_2(x,\gamma_m) \tag{27}$$

are given by

$$\varphi_m^{(2)}=\frac{D_2(\gamma_m)}{\dot{D}_1(\gamma_m)E(\gamma_m)}\sum_{m=1}^{\infty}\psi_n^{(1)}\frac{S_1(x_c,\gamma_m)S_1'(x_c,\beta_n)-S_1'(x_c,\gamma_m)S_1(x_c,\beta_n)}{\beta_n-\gamma_m}, \tag{28}$$

where $\dot{D}_1(\gamma_m)$ denotes $(dD_1(\lambda)/d\lambda)_{\lambda=\gamma m}$. Hence the element of the matrix B_{21} at the m-th row and the n-th column $B_{21,mn}$ may be written as follows.

$$B_{21,mn}=\frac{D_2(\gamma_m)}{\dot{D}_1(\gamma_m)E(\gamma_m)}\,\frac{S_1(x_c,\gamma_m)S_1'(x_c,\beta_n)-S_1'(x_c,\gamma_m)S_1(x_c,\beta_n)}{\beta_n-\gamma_m}. \tag{29}$$

There are other ways to express $W^{(2)}(x)$ as a vector. For instance, γ_m's may be taken to be zeros of $S_2(x,\lambda)$. To obtain the matrix B_{21}, it is only necessary to expand $W^{(2)}(x)$ given by (26) into a series in the interval (x_c, x_2). But in general, the calculation of the elements involves matrix multiplication.

The elements of the matrix A_{12} may also be calculated by the similar arguments in which $D_1(\lambda)$ and $D_2(\lambda)$ are interchanged.

6. Conclusion

By the aid of integral transforms, the functional equations appearing in mixed boundary value problems in a complicated domain may be expressed as matrix equations so that the solution may be obtained by matrix inversion.

References

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