Solution of Vlasov Equation

Tomejiro YAMAGISHI

Abstract

A semi group operator is introduced to solve the time-dependent Vlasov equation in a general coordinate system. The method is applied for derivation of liner perturbation of plasma distribution function in the electrostatic approximation.

§ 1 Introduction

The Vlasov equation, which is the Boltzmann transport equation without particle-partilce collision term, describes the distribution of charged as well as neutral particles in the configuration and velocity spaces. It is used to describe microscopic behavior of plasma, i.e., in the plasma kinetic theory. The velocity moments of the Vlasov equation yield fluid model equatuions. Therefore, the Vlasov equation is the basic equation for plasma stability as well as transport theories. The solution of Vlasov equation has the singurality due to the wave-particle resonance, and therefore it is not square integrable (not involved in the Hilbert space). The eigenvalue spectrum always involves either one or two dimensional continuum depending on the model. The complete set of eigenfunctions is constructed in a class of generalized function space[1),2),3),4)].

In this report, we consider a different aspect of the equation, i.e., the time evolution operator (semi group) of the Vlasov equation.

§ 2 Vlasov equation

The Vlasov equation for the particle distribution function f (**x**, **v**, t) is decribed in a simple form :

$$\frac{\partial f}{\partial t}+Lf=0 \tag{1}$$

where L is the differential operator defined by

$$L\equiv \mathbf{v}.\ \nabla+\frac{q}{m}(\mathbf{E}+\frac{1}{c}\ \mathbf{v}\times\mathbf{B})\frac{\partial}{\partial \mathbf{v}} \tag{2}$$

Notations **x**, **v** represent configuration and velocity spaces, q and m are charge and mass, **E** and **B** are electric and magnetic fields, respectively, and other notations are standard.

Since Eq.(1) is the first order differential equation with respect to time t, the solution can be written as

$$f\ (\mathbf{x},\ \mathbf{v},\ t)=\exp\ (\int dt\ L)\ f_0\ (\mathbf{x},\ \mathbf{v}) \tag{3}$$

where f_0 is the initial distribution function which may be obtained from the collisional relaxation theory, and is usually approximated by the Maxwellian distribution function. The operator exp ($\int$ dt L) in Eq.(3) describes the time evolution of the solution, and can be considered as a semi group operator, which has also been used in the problem of electromagnetic anomalous transport problem[5),6)]. Since this operator consists of the differantial operator ∇ and $\partial/\partial\mathbf{v}$ in the configuration and velocity spaces, Eq.(3) may be written in the form

$$f\ (\mathbf{x},\ \mathbf{v},\ t)=f_0\ (\mathbf{x}+\tilde{\mathbf{x}},\ \mathbf{v}+\tilde{\mathbf{v}}) \tag{4}$$

where the displacements are given by $\tilde{\mathbf{x}}=\int dt\mathbf{v}$ and $\tilde{\mathbf{v}}=\int dtq\ (E+v\times B/c)/m$, respectively. This means that the solution operator in Eq.(3) just gives the displacements $\tilde{\mathbf{x}}$, $\tilde{\mathbf{v}}$ in the initial distribution function.

We have no further information about the solution in this Lagrangean approach. In what follows, in order to obtain useful form of the solution, we will employ the perturbation method.

We decompose f and other related quantities into the ensemble averaged (equilibrium) $\bar{f}$ and small fluctuation $\tilde{f}$ as in usual manner : $f=\bar{f}+\tilde{f}$ where the bar over f represents the ensemble average. The operator L can also be decomposed as $L=\bar{L}+\tilde{L}$, where

$$\bar{L}=\mathbf{v}.\ \nabla-(\frac{q}{m}\nabla\bar{\phi}-\Omega\mathbf{v}\times\mathbf{b})\frac{\partial}{\partial\mathbf{v}} \tag{5}$$

$$\tilde{L}=-\frac{q}{m}\nabla\tilde{\phi}\ \frac{\partial}{\partial\mathbf{v}} \tag{6}$$

Here the electric field has been expressed in term of the scalar potential $\mathbf{E}=-\nabla\bar{\phi}-\nabla\tilde{\phi}$, $\mathbf{b}=\mathbf{B}/B$, and the magnetic fluctuation has been neglected. Introducing these decompositions into Eq.(1), we have

$$\frac{\partial\bar{f}}{\partial t}+\frac{\partial\tilde{f}}{\partial t}+\bar{L}\bar{f}+\tilde{L}\tilde{f}+\bar{L}\tilde{f}+\tilde{L}\tilde{F}=0. \tag{7}$$

Since $\bar{\tilde{f}}=0$ and $\bar{\bar{f}}=\bar{f}$, the ensemble average of Eq.(7) yields

$$\frac{\partial\bar{f}}{\partial t}+\bar{L}\bar{f}=-\overline{\tilde{L}\tilde{f}}. \tag{8}$$

The right hand side of Eq.(8) represents the nonlinear effect which involves anomalous transports induced by the fluctuations both in the configuration and velocity spaces[7]. Subtracting the both sides of Eq.(8) from Eq.(7), we have

$$\frac{\partial \tilde{f}}{\partial t}+\bar{L}\tilde{f}+\tilde{L}\tilde{f}-\overline{\tilde{L}\tilde{f}}=-\tilde{L}\bar{f} \tag{9}$$

which governs the fluctuation $\tilde{f}$. The third and fourth terms in the left side of Eq.(9) represents the nonlinear effects. In the following, we consider the linear portion of Eq. (9) neglecting these nonlear terms.

§ 3 Operational method

3. 1 Equilibrium solution

We first consider the averaged solution $\bar{f}$ which satisfies Eq.(8). The equilibrium solution $\bar{f}$ has slow time as well as spatial variations due to the dissipations. In the microscopic time scale, the slow time variation due to transports may be neglected. If we neglect the nonlinear dissipation term in Eq.(8), the solution of Eq.(8)may be given in the form of Eq.(3) by replacing L by $\bar{L}$. Without the dissipations in Eq.(8), the time variation may be small scale Larmor gyromotion. We therefore relate time t with the phase angle ζ of particle gyromotion by $t=\zeta/\Omega$ where Ω is the Larmor frequency. If we expand the semi group operator in Eq.(3), we have

$$\bar{f}\ (\mathbf{x},\ \mathbf{v})=\sum_{n=0}^{\infty} f_n\ (\mathbf{x},\ \mathbf{v}) \tag{10}$$

where

$$f_n\ (\mathbf{x},\ \mathbf{v})=\frac{1}{n\ !}D^n\ f_0\ (\mathbf{x},\ \mathrm{v}) \tag{11}$$

and $D=\int d\zeta\ (\mathbf{v}.\ \nabla-(q/m)\ \nabla\bar{\phi}\ \partial/\partial\ \mathrm{v})/\Omega$. With this operator, $\bar{f}$ is equivalent to the exact solution which satisfies $\bar{\bar{L}}f=0$. When the lowest order solution f_0 is given, the equilibrium solution is obtained iteratively : $f_n=Df_{n-1}/n$. The solution may be approximated by a small number of expansion terms when the norm $\|D\|$ is much smaller than unity, which holds in usual situations. Rutherford et al[8]. solved the equation $\bar{\bar{L}}f=0$ iteratively. Our simple solution involves all order of perturbations.

Let us derive a concrete form of solution. In velocity space, we write $\mathbf{v}=\mathbf{v}_\perp+\mathbf{v}_\parallel\mathbf{b}$, and use as coordinates the energy per uint mass $E=v^2/2+q\bar{\phi}/m$ and the magnetic moment $\mu=v_\perp^2/(2B)$. The perpendicular component of velocity vector can be written by $\mathrm{v}_\perp=\mathrm{v}_\perp\cdot(\mathrm{e}_1\cos\zeta+\mathrm{e}_2\sin\zeta)$, where the orthogonal unit vectors e_1 and e_2 are perpendicular to the magnetic field line $\mathbf{b}$. In the transformation of coordinate $(\mathbf{x},\ \mathrm{v})\rightarrow(\mathrm{x}',\ E,\ \mu,\ \zeta)$, the derivatives in the operator $\bar{L}$ may be written in the form[8] :

$$\nabla = \nabla' + \frac{q}{m} \nabla \bar{\phi} \frac{\partial}{\partial \varepsilon} - (\mu \nabla B + \upsilon_{11} (\nabla \mathbf{b}) \mathbf{v}_{\perp}) \frac{1}{B} \frac{\partial}{\partial \mu}$$

$$+ [(\nabla \mathbf{e}_2) \mathbf{e}_1 + \frac{\upsilon_{11}}{\upsilon_{\perp}} (\nabla \mathbf{b})(\mathbf{v}_{\perp} \times \mathbf{b})] \frac{\partial}{\partial \zeta} \tag{12}$$

$$\frac{\partial}{\partial \mathbf{v}} = \mathbf{v} \frac{\partial}{\partial \varepsilon} + \frac{\mathbf{v}_{\perp}}{B} \frac{\partial}{\partial \mu} - \frac{\mathbf{v} \times \mathbf{b}}{\upsilon_{\perp}^2} \frac{\partial}{\partial \zeta} \tag{13}$$

where $v_{11}^2 = 2\ (\varepsilon - \mu B - q\bar{\phi}/m)$. Making use of these relations (12) and (13), we have

$$D = \frac{\mathbf{v} \times \mathbf{b}}{\Omega} \nabla' - [\mathbf{v}_{\perp} \mathbf{v}_D + \frac{\upsilon_{11}}{\Omega} \int d\zeta\ (\mathbf{v}_{\perp} \mathbf{v}_{\perp} : \nabla \mathbf{b} - \frac{\upsilon_{\perp}^2}{2} \nabla . \mathbf{b})] \frac{1}{B} \frac{\partial}{\partial \mu} \tag{14}$$

where v_D is the precessional drift velocity due to the inhomogenieties of magnetic field and electric potential : $v_D = \mathbf{b} x\ (\mu \nabla B + v_{11}^2 \kappa + q \nabla \bar{\phi}/m)/\Omega$, and $\kappa = (\mathbf{b}\ . \nabla)\ \mathbf{b}$ is the curvature of magnetic field line. As we shall see, the first term in Eq.(14), when operated to f_0, reduces to the diamagnetic drift term. The equilibrium solution may be approximated by $\bar{f} = f_0 + Df_0$[8].

3．2 Solution for perturbation

We now proceed to derivation of the solution $\tilde{f}$ of Eq.(9) neglecting the nonlinear (third and fourth) terms. Since Eq.(9) is also a first order differential equation, we obtain

$$\tilde{f} = -\int_{\infty}^{t} dt' \exp\ (\int_{t}^{t'} dt\bar{L})\ \tilde{L}\bar{f}, \tag{15}$$

In order to have familiar form of solution, we apply the eikonal representation for the perturbation : $\tilde{\phi} = \hat{\phi} \exp\ (iS - i\omega t)$, where the function S, which must be invariant on the magnetic surface and satisfies $\mathbf{b}.\ \nabla S = 0$, may be given by $S = -n\ (\theta - \int \nu\, d\chi)$ with $\nu = JB/R$ being the inverse rotational transform, n is the toroidal mode number in the flux coordinate system $(\psi,\ \chi,\ \theta)$[9]. It is also assumed that $(\nabla S \gg \nabla \hat{\phi} / \hat{\phi}$ for $n \gg 1$.

We first consider the source term $\tilde{L}\ \bar{f}$ in Eq.(15). Making use of approximation $\bar{f} = f_0 + Df_0$, and Eqs.(12), (13) and (14), we have

$$\nabla \tilde{\phi} \frac{\partial \bar{f}}{\partial \mathbf{v}} = \mathbf{v} . \nabla \tilde{\phi}\ (\frac{\partial f_0}{\partial \varepsilon} + \frac{1}{B} \frac{\partial f_0}{\partial \mu}) - \upsilon_{11} \mathbf{b}\ . \nabla \tilde{\phi} \frac{1}{B} \frac{\partial f_0}{\partial \mu}$$

$$- i\ \tilde{\phi}\ [\nabla S \frac{\mathbf{v} \times \mathbf{b}}{\upsilon_{\perp}^2} \frac{\partial}{\partial \zeta} (\frac{\mathbf{v} \times \mathbf{b}}{\Omega})\ \nabla f_0 - \frac{\omega_D}{B} \frac{\partial f_0}{\partial \mu}] \tag{16}$$

where $\omega_D \equiv v_D .\ \nabla S$ is the precessional drift frequency. The third term in Eq.(16), which comes from $\partial D / \partial \mathbf{v}$, gives the diamagnetic drift term when it is averaged over ζ. In the derivation of Eq.(16), $\nabla\tilde{\phi} D\partial f_0/\partial \mathbf{v}$ has been neglected, because it has second order derivaties. If we apply the relation[10] $\mathbf{v}.\ \nabla = d/dt - \partial/\partial t$ in the first term in Eq.(16), and introduce into Eq.(15), we obtain

$$\tilde{f} = \frac{q}{m}\tilde{\phi}\left(\frac{\partial f_0}{\partial \varepsilon} + \frac{1}{B}\frac{\partial f_0}{\partial \mu}\right) - i\frac{q}{m}\int_{\infty}^{t} dt' \exp\left(\int_{t}^{t'} dt'' \bar{L}\right)\left[\omega\tilde{\phi}\frac{\partial f_0}{\partial \varepsilon}\right.$$

$$\left. + (\omega + \omega_D - i\, v_{11}\mathbf{b}\nabla)\frac{\tilde{\phi}}{B}\frac{\partial f_0}{\partial \mu} + \tilde{\phi}\frac{\nabla S.\ (\mathbf{v}\times\mathbf{b})}{v_{\perp}^2\Omega}\mathbf{v}_{\perp}.\ \nabla f_0\right] \tag{17}$$

We now examine some characteristics of the semi group operator in Eq.(17), which is essentially the same as the propagator used in a previous paper[11]. We decompose the velocity in $\bar{L}$ defined by Eq.(5) into the perpendicular ($\mathbf{v}_{\perp}$) and parallel ($v_{11}\mathbf{b}$) components, and decompose $\mathbf{v}_{\perp}$ again into the Larmor gyromotion component and slow precessional drift velocity $\mathbf{v}_D$. Making use of the relation $\mathbf{v}\times\mathbf{b}\partial/\partial\mathbf{v} = -\partial/\partial\zeta$ from Eq.(13), we have

$$\int_{t}^{t'} dt\bar{L} = i\ (G(\zeta') - G(\zeta)) + (i\,\omega_D + v_{11}\mathbf{b}\ .\mathbf{v})\ \tau + \tilde{\mathbf{v}}\frac{\partial}{\partial \mathbf{v}} - \Omega\tau\frac{\partial}{\partial \zeta} \tag{18}$$

where $G(\zeta) \equiv \int d\zeta\, \mathbf{v}_{\perp}.\ \nabla S/\Omega = \mathbf{v}\times\mathbf{b}\ .\nabla S/\Omega$ represents the Larmor gyromotion effect, $\tilde{\mathbf{v}} = -q\int dt\nabla\tilde{\phi}/m$ is the velocity induced by the electric field and $\tau = t' - t$. When we introduce Eq.(18) into the semi group operator in Eq.(17), the third and fourth terms in Eq.(18) give the displacements $\tilde{\mathbf{v}}$ and $-\Omega\tau$ in $\mathbf{v}$ and ζ coordinates on the operand in Eq.(18). We neglect, however, these displacements, because $\tilde{\mathbf{v}}$ is much smaller than particle velocity $\mathbf{v}$ and f_0 is independent of ζ.

When we express $G = a\sin\zeta$ with $a = \mathbf{v}_{\perp}.\ \nabla S/\Omega$, and use the formula $\exp(ia\sin\zeta) = \Sigma J_l(a)\exp(il\zeta)$, we have

$$\exp\left(\int_{t}^{t'} dt.\ \bar{L}\right) = \exp\{(i\omega_D + v_{11}\mathbf{b}\ .\nabla)(t'-t)\}\ \sum_{l,l'} J_l(a)J_{l'}(a)\exp i(l\zeta' - l'\zeta) \tag{19}$$

Substituting Eq.(19) into Eq.(17), and carrying out the time integral by τ bearing in mind that $\zeta(t') = \zeta(t) + \Omega\tau$, we obtain

$$\hat{f} = \frac{q}{m}\hat{\phi}\left(\frac{\partial f_0}{\partial \varepsilon} + \frac{1}{B}\frac{\partial f_0}{\partial \mu}\right) - \frac{q}{m}\sum_{l,l'} J_l(a)J_{l'}(a)\mathrm{esp}\,i(l-l)\zeta$$

$$(\omega + \omega_D + i\,v_{11}\mathbf{b}\nabla + l\Omega)^{-1}\left[(\omega + \omega_D + i\,v_{11}\mathbf{b}\ .\mathbf{v})\frac{\hat{\phi}}{B}\frac{\partial f_0}{\partial \mu}\right.$$

$$\left. + \hat{\phi}\left(\omega\frac{\partial f_0}{\partial \varepsilon} - \frac{\mathbf{b}.\ \mathbf{v}\times\nabla S}{v_{\perp}^2}\cdot\frac{\mathbf{v}_{\perp}\nabla f_0}{\Omega}\right)\right] \tag{20}$$

If we consider the low frequency regime, $\omega \ll \Omega$, the $l = 0$ contribution becomes dominant, and we have additional term which does not have ω from the second term in Eq.(29) :

$$\hat{f}=\frac{q}{m}\hat{\phi}\left\{\frac{\partial f_0}{\partial \varepsilon}+\left(1-\sum_l J_l(a)J_0(a)e^{-ils}\right)\frac{1}{B}\frac{\partial f_0}{\partial \mu}\right\}-\frac{q}{m}\sum_l J_0 J_l e^{-ils}\times$$

$$(\omega+\omega_D+i\,\upsilon_{11}\mathbf{b}\,.\nabla)^{-1}\left(\omega\frac{\partial f_0}{\partial \varepsilon}-\frac{\mathbf{b}.\ \mathbf{v}\times Ds}{\upsilon_\perp^2}\ \frac{v_\perp \nabla f_0}{\Omega}\right) \tag{21}$$

When Eq.(21) is averaged over ζ, it reduces to the result derived from the gyrokinetic theory[8]. The second term in Eq.(21) corresponds to the function h[8]. The result (21) is a basis of plasma kinetic theory. Plasma stabilities may be studied by integrating Eq.(21) over velocity space for both electrons and ions, and introducing into the neutrality condition.

§ 4 Summary

A semi group operator is derived for the plasma Vlasov equation in the electrostatic approximation. It is applied for dervation of kinetic solution with the eikonal representation in a general coordnate system. Our kintic solution is found to reduce the gyro-kinetic solution in the low frequency limit by averaging over the gyro-phase angle.

References

1) N. G. Van Kampen, Physica 21, 945 (1955).
2) K. M. Case and J. H. Zweifel, *Linear Transport Theory*, Addisonwesely, New York, 1967.
3) T. Yamagishi, Technol. Rep. Osaka Univ., 19, 301 (1969).
4) T. Yamagishi, Transport Theory and Statistical Physics 2, 107 (1973).
5) J. Weinstock, Phys. Fluids 12, 1045 (1969).
6) T. Yamagishi, Phys. Fluids 22, 2431 (1979).
7) T. Yamagishi, Plasma Physics and Controlled Fusion 28, 453 (1986).
8) P. H. Rutherford and E. A. Frieman, Phys. Fluids 11, 569 (1968).
9) J. W. Conner, R. J. Hastie, and J. B. Taylor, Phys. Rev. Lett. 40, 396(1978), and Proc. R. Soc. London Ser. A 365, 1 (1979).
10) M. N. Rosenbluth, N. A. Krall and N. Rostoker, Nucl. Fusion Suppl. Pt. 1, 143 (1962).
11) T. Yamagishi, J. Phys. Soc. Japan 57, 2732 (1988).