

A Charge Simulation Method for the Calculation of Two-dimensional Electrostatic Fields

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Abstract—A charge simulation method which is a numerical method for the computation of electrostatic fields is described. Two modified methods of the charge simulation method, i. e., the multiple charges method and the charge compression method, are proposed for improving the accuracy of computation. Two basic methods and two modified methods are applied to solve a two-dimensional field problem satisfying Laplace's equation, and their computation errors are discussed.

1. Introduction

The calculation of electric fields requires the solution of Laplace's and Poisson's equation with boundary conditions satisfied. This can be done either by analytical or numerical methods. In many instances, physical systems are so complex that analytical solutions are difficult or impossible, and hence numerical methods are commonly used for engineering applications [1], [2].

The available numerical methods are all based on the discrete division of the continuity and can be classified as follows [3] :

(1) Division of the whole field region :

(1a) Division with lattices : FINITE DIFFERENCE METHOD [1],

(1b) Division with finite elements : FINITE ELEMENT METHOD.

(2) Division of the boundary region and charges :

(2a) Use of discrete fictitious charges placed outside the field space, i. e., inside the electrodes : CHARGE SIMULATION METHOD [2], [4],

(2b) Use of surface charges : SURFACE CHARGE METHOD.

This paper describes the principle and applications of the charge simulation method. Two modified methods of the charge simulation method are proposed for improving the accuracy of computation with a small number of fictitious charges. One is the multiple charges method and the other is the charge compression method [5], [6]. Two basic methods and two modified methods are applied to solve a two-dimensional field problem satisfying Laplace's

equation, and their computation errors are discussed in the relation of the assignment factor.

2. Laplace's Equation and Model Problem

In this section, a two-dimensional field problem to be solved numerically in section 3 is shown and its analytical solution is also presented.

2.1 Laplace's equation

The calculation of the electrostatic field is reduced to the boundary-value problem of the first kind, i. e., Dirichlet problem, where (1) Laplace's equation is satisfied in the region S under consideration :

$$\nabla^2 \phi(s) = 0, \quad s \in S, \quad (1)$$

and (2) the boundary condition is satisfied on the boundary Γ :

$$\phi(s) = f(s), \quad s \in \Gamma, \quad (2)$$

where ∇^2 is the Laplace operator and s is the location. In the electrostatic field problem, ϕ is generally taken as the electric potential and Γ is generally taken as the surfaces of electrodes.

For the two-dimensional field, Laplace's equation (1) is described as

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \quad (3)$$

2.2 Model problem

Fig. 1 shows a model problem in the two-dimensional square region. Laplace's equation (3) is satisfied in the region. The boundary conditions are as follows :

$$\left. \begin{aligned} \phi &= U \cos \frac{\pi x}{a} \text{ [V]} \quad \text{for } -\frac{a}{2} \leq x \leq \frac{a}{2}, \quad y = b, \\ \phi &= 0 \text{ [V]} \quad \text{for } x = \pm \frac{a}{2}, \quad 0 \leq y \leq b, \quad \text{and for } -\frac{a}{2} \leq x \leq \frac{a}{2}, \quad y = 0. \end{aligned} \right\} \quad (4)$$

This model problem will be solved numerically by means of the charge simulation method in section 3.

The analytical solution $\phi(x, y)$ is obtained by the method of separation of variables [1], [7] :

$$\phi(x, y) = U \cos \frac{\pi x}{a} \frac{\sinh \frac{\pi y}{a}}{\sinh \frac{\pi b}{a}} \text{ [V]}. \quad (5)$$

Substituting

$$a = b = 8 \text{ [m]} \quad \text{and} \quad U = 1 \text{ [V]}, \quad (6)$$

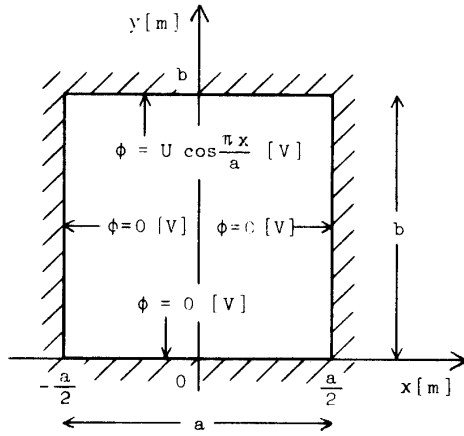


Fig. 1. Model problem in the two-dimensional square region, where $a=b=8$ [m] and $U=1$ [V].

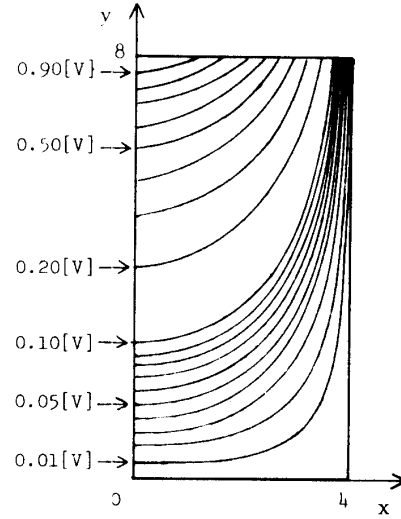


Fig. 2. Potential distribution.

the following expression is obtained :

$$\phi(x, y) = \cos \frac{\pi x}{8} \frac{\sinh \frac{\pi y}{8}}{\sinh \pi} \quad [\text{V}]. \quad (7)$$

Fig. 2 shows the potential distribution for Fig. 1.

3. Charge Simulation Method

In this section, two modified methods of the charge simulation method, i. e., the multiple charges method and the charge compression method, are proposed [5], [6]. These modified methods can improve the accuracy of the field calculation using a small number of fictitious charges. Two basic methods and two modified methods are applied to solve the model problem of Fig. 1, and their computation errors are investigated in the relation of the assignment factor f_a .

3.1 Principle

The numerical application of the charge simulation method was developed by Steinbigler in 1969 [4], and proved to be successful for many high-voltage field problems. It is very simple and it is applicable to any system including one or more homogeneous media. This numerical method is based on the integral concept.

In this method, the potentials of fictitious charges are taken as particular solutions of

Laplace's and Poisson's equations. Physically the distributed surface charges are replaced by discrete fictitious charges. These charges are placed outside the space in which the field is to be computed. The magnitudes of these charges have to be calculated so that their integrated effect satisfies the boundary conditions exactly at a selected number of points on the boundary. As the potentials due to these charges satisfy Laplace's or Poisson's equation inside the space under consideration, the solution is unique inside that space.

The procedure of applying the charge simulation method to the high-voltage field calculation is as follows [5], [8] :

(Step 1) Place n fictitious charges outside the field space, i. e., inside the electrodes, as shown in Fig. 3.

(Step 2) Place n contour points on the boundary, i. e., on the surfaces of electrodes.

(Step 3) Place m ($\cong n$) check points on the boundary.

(Step 4) Solve n linear equations :

$$[P_{ij}] [Q_j] = [\phi_i], \quad (8)$$

where P_{ij} : the potential coefficient between the i -th contour point and the j -th fictitious charge, including the effect of the corresponding image charge; known value,

Q_j : the j -th charge; unknown value,

ϕ_i : the potential at the i -th contour point; known value.

(Step 5) Compute the potential ϕ at each check point (the index i for the contour point is neglected) :

$$\phi = \sum_{j=1}^n P_j Q_j, \quad (9)$$

where P_j : the potential coefficient between the check point and the j -th charge,

and compute the difference between the computed potential ϕ and the true potential given by the boundary conditions.

(Step 6) If the differences are large, change the location and/or the number of the fictitious charges, and continue Step 1 — Step 5.

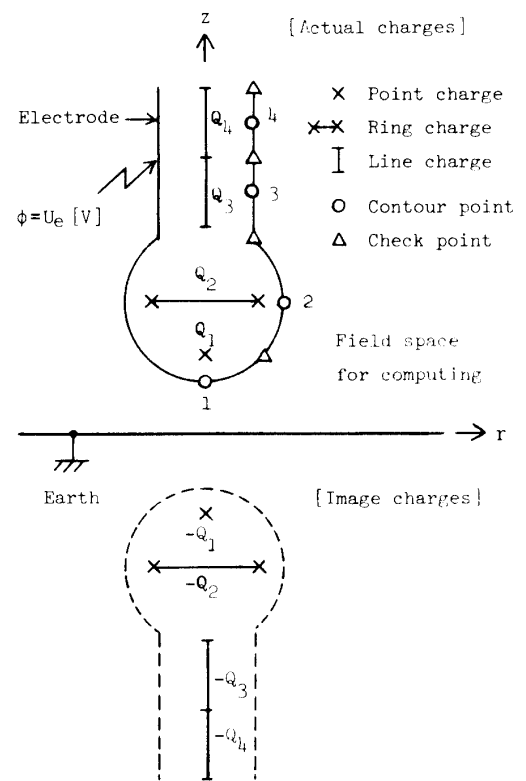


Fig. 3. Principle of the charge simulation method: An example of the three-dimensional field with axial symmetry.

(Step 7) Compute the potential ϕ and/or the field strength E_r , E_z at the optional point (r, z) in the field space :

$$\phi(r, z) = \sum_{j=1}^n P_j Q_j, \quad (10)$$

$$E_r(r, z) = \sum_{j=1}^n F_{rj} Q_j, \quad (11)$$

$$E_z(r, z) = \sum_{j=1}^n F_{zj} Q_j, \quad (12)$$

where E_r , E_z : the r – and z –components of the field strength,

F_{rj} , F_{zj} : the coefficients for E_r , E_z .

The charge simulation method has the following advantages in comparison with the finite difference method and the finite element method [2], [3], [8] :

- (1) It is not necessary that the field region is limited by a closed boundary.
- (2) The field strength can be calculated analytically by using the computed charges.
- (3) The computation times are short in many geometries used in high-voltage technology.
- (4) Constructing high-voltage apparatus one generally prefers curved surfaces to sharp edges; therefore in that case the method is well usable.
- (5) The method shows the good applicability to three-dimensional fields without axial symmetry and to space charge problems.

The charge simulation method has the following disadvantages [3], [8] :

- (1) It is complex to apply to the field calculation of an arrangement including corners and/or edges (e. g. transformer).
- (2) The application for much more than two dielectrics is possible in principle, but the computer expense increases extremely.

3.2 Two-dimensional electrostatic fields

The two-dimensional electrostatic fields are calculated by the use of infinite line charges perpendicular to the x – y plane. The potential coefficients of infinite line charges are defined by the expression [2], [4] :

$$P_j = \frac{1}{2\pi\epsilon} \ln \frac{\sqrt{(x-x_j)^2 + (y+y_j)^2}}{\sqrt{(x-x_j)^2 + (y-y_j)^2}} \quad [\text{m/F}], \quad (13)$$

where ϵ is the permittivity, and (x_j, y_j) are the coordinates of the j -th infinite line charge, and (x, y) are the coordinates of the optional point for computing. This expression is derived from Gauss' theorem and the physical condition that the earthed infinite plane at $y=0$ keeps at zero potential [7]. Eq. (13) corresponds to Eqs. (8)–(10), and includes the parts of image charges for the representation of the earthed infinite plane.

Since the line charges are of infinite length, the quantities to be determined in Eq. (8) are charges per unit length Q_j [C/m].

After the check of the boundary conditions (Step 5, Step 6), the x - and y -components of the field strength at any point (x, y) can be calculated by means of the following expressions [2], [4] :

$$E_x = \sum_{j=1}^n \frac{Q_j}{2\pi\epsilon} \left\{ \frac{x-x_j}{(x-x_j)^2 + (y-y_j)^2} - \frac{x-x_j}{(x-x_j)^2 + (y+y_j)^2} \right\} \quad [\text{V/m}], \quad (14)$$

$$E_y = \sum_{j=1}^n \frac{Q_j}{2\pi\epsilon} \left\{ \frac{y-y_j}{(x-x_j)^2 + (y-y_j)^2} - \frac{y+y_j}{(x-x_j)^2 + (y+y_j)^2} \right\} \quad [\text{V/m}]. \quad (15)$$

Eqs. (14) and (15) correspond to Eqs. (11) and (12).

3.3 Applications of the charge simulation method to the model problem

Fig. 4 shows the arrangements of fictitious charges and contour points for solving the model problem of Fig. 1 with three types of charge simulation method. In Fig. 4, the arrangement is shown only in the right half plane, because the model problem of Fig. 1 is symmetric with respect to the y -axis. Therefore, for example, the arrangement shown in Fig. 4(a) is equivalent to the complete arrangement shown in Fig. 5.

In the case of the basic method shown in Fig. 4(a), contour points are arranged at equal intervals on the boundary. In the case of the multiple charges method shown in Fig. 4(b), contour points are arranged at equal intervals on the boundary, and five multiple charges Q_{12} correspond to the twelfth contour point. In the case of the charge compression method shown in Fig. 4(c), contour points are not arranged at the equal intervals on the boundary. The density of the contour points is increased around the upper corner where the electrostatic field varies greatly, and it is decreased at the place away from the corner.

Because the number of contour points, i. e., the number of fictitious charges, is $n=12$ in any of Figs. 4(a)–(c), the number of unknown variables in Eq. (8) is also $n=12$ for any type of charge simulation method.

In the case of Figs. 4(a) and (c), the potential ϕ at the point (x, y) is expressed by the relations :

$$\phi(x, y) = \frac{1}{4\pi\epsilon} \left[Q_1 \ln \frac{\{(x-x_1)^2 + (y+y_1)^2\}}{\{(x-x_1)^2 + (y-y_1)^2\}} + \sum_{j=2}^{12} Q_j \ln \frac{\{(x-x_j)^2 + (y+y_j)^2\} \{(x+x_j)^2 + (y+y_j)^2\}}{\{(x-x_j)^2 + (y-y_j)^2\} \{(x+x_j)^2 + (y-y_j)^2\}} \right] \quad [\text{V}], \quad (16)$$

$$\frac{1}{4\pi\epsilon} = 8.988 \times 10^9 \quad [\text{m/F}], \quad (17)$$

where (x_j, y_j) are the coordinates of the fictitious charge Q_j . In the case of Fig. 4(b), the potential ϕ is expressed by the relations :

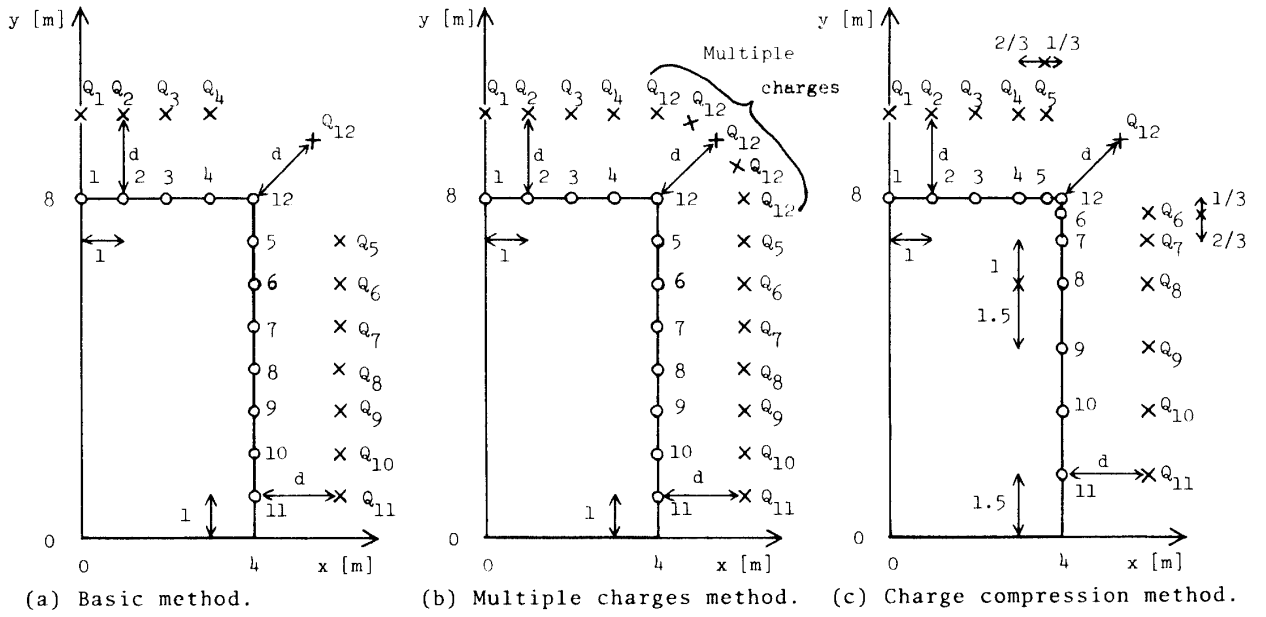


Fig. 4. Arrangements of fictitious charges (x) and contour points (o).

$$\phi(x, y) = \frac{1}{4\pi\epsilon} \left[\left\{ \text{the same expression in Eq. (16), for } 1 \leq j \leq 11 \right\} + Q_{12} \sum_{k=1}^5 \ln \frac{\{(x-x_{12,k})^2 + (y+y_{12,k})^2\} \{(x+x_{12,k})^2 + (y+y_{12,k})^2\}}{\{(x-x_{12,k})^2 + (y-y_{12,k})^2\} \{(x+x_{12,k})^2 + (y-y_{12,k})^2\}} \right] \text{ [V], (18)}$$

$$\left. \begin{aligned} \text{where } x_{12,k} &= 4 + d \cos \frac{(k-1)\pi}{8}, \\ y_{12,k} &= 8 + d \sin \frac{(k-1)\pi}{8}. \end{aligned} \right\} \quad (19)$$

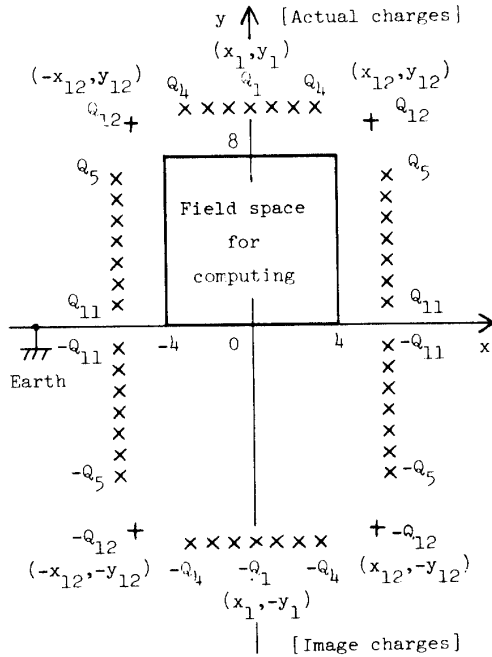


Fig. 5. Complete arrangement of fictitious charges for Fig. 4 (a).

For the most effective application of the charge simulation method, the question of a suitable arrangement of the charges and contour points is of importance. A practical criterion is obtained by the definition of an assignment factor :

$$fa \triangleq a_2/a_1 \quad (20)$$

with the distance a_1 [m] between two successive contour points and the distance a_2 [m] between a contour point and its corresponding charge [2]. For the examples of Fig. 4, the following relations are used :

$$a_1 = 1 \text{ [m]}, \quad a_2 = d \text{ [m]}, \quad \text{and } fa = d. \quad (21)$$

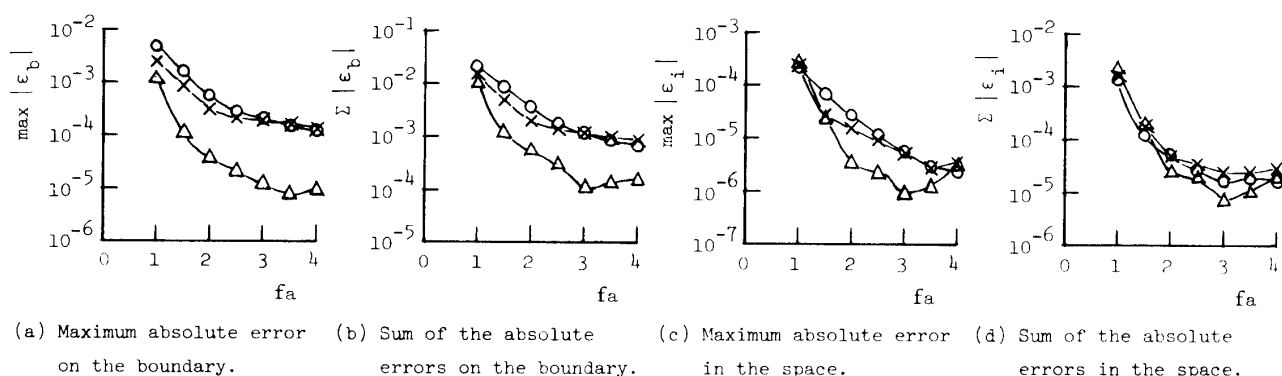


Fig.6. Computation errors versus assignment factor f_a .

(○ : Basic method, × : Multiple charges method, △ : Charge compression method)

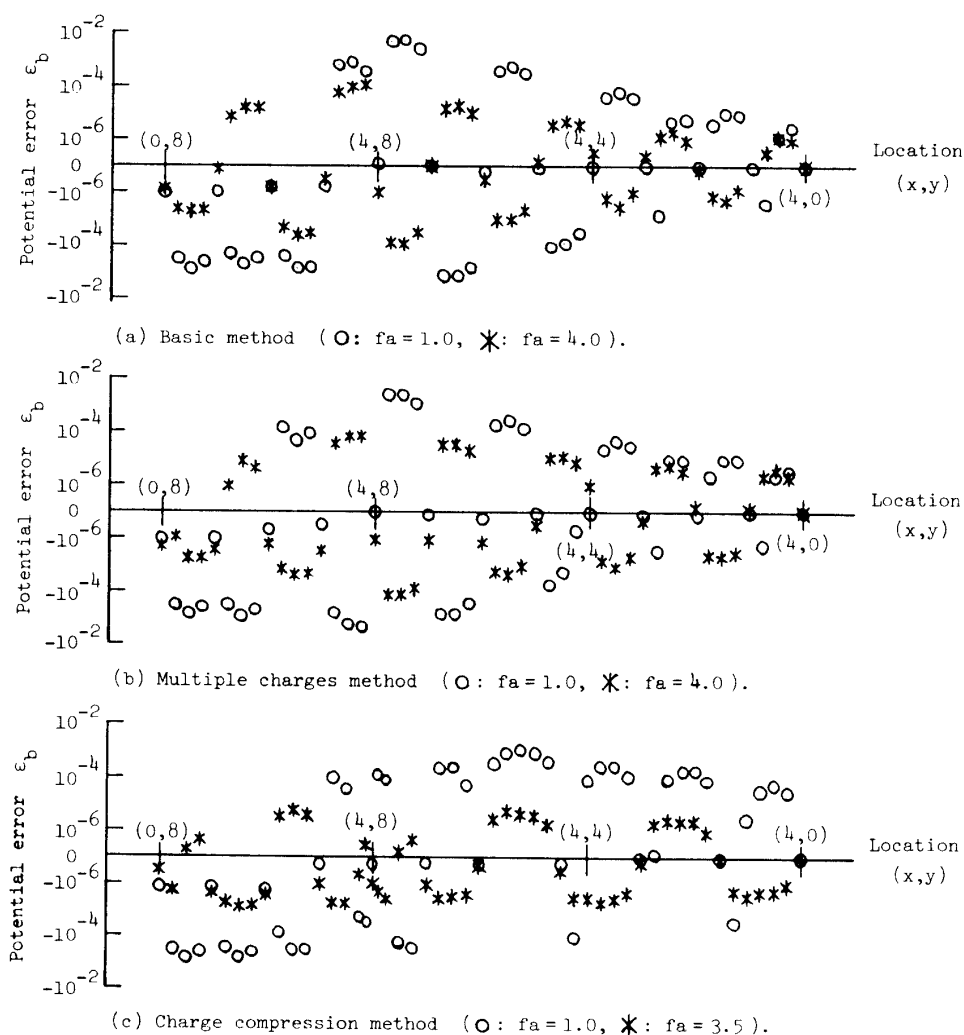


Fig. 7. Potential errors on the boundary, where the vertical axes are in logarithmic scale for $|\epsilon_b| \geq 10^{-6}$ and in normal scale for $|\epsilon_b| \leq 10^{-6}$.

The accuracy of the computation depends on the choice of the assignment factor as shown in Fig. 6. The potential error at the point (x, y) is defined as :

$$\text{Potential error} \triangleq \phi_{\text{calculated}}(x, y) - \phi_{\text{exact}}(x, y), \quad (22)$$

where $\phi_{\text{calculated}}$ is the value of potential calculated by the charge simulation method, and ϕ_{exact} is the exact value of potential given by Eq. (7). The potential error ε_b is evaluated on the boundary $(0, 8) \rightarrow (4, 8) \rightarrow (4, 0)$ by the step length 0.25 [m]. The potential error ε_i is evaluated in the space $0 \leq x \leq 3, 0 \leq y \leq 7$ by the step length 1.0 [m].

In the high-voltage technology, it is required that the computation errors of the potential and the field strength should be less than 0.1 [%] and 1.0 [%] respectively for high quality calculation [2]. This requirement is satisfied sufficiently as shown in Figs. 6(a), (c) and 7. That is,

- (a) Basic method (of $n=12$) : $\max |\varepsilon_b| = 0.119 \times 10^{-3}, fa=4.0, x=3.75, y=8.0,$
 $\max |\varepsilon_i| = 0.265 \times 10^{-5}, fa=4.0, x=3.0, y=7.0,$
- (b) Multiple charges method : $\max |\varepsilon_b| = 0.126 \times 10^{-3}, fa=4.0, x=4.0, y=7.75,$
 $\max |\varepsilon_i| = 0.283 \times 10^{-5}, fa=3.5, x=3.0, y=7.0,$
- (c) Charge compression method : $\max |\varepsilon_b| = 0.805 \times 10^{-5}, fa=3.5, x=1.5, y=8.0,$
 $\max |\varepsilon_i| = 0.970 \times 10^{-6}, fa=3.0, x=1.0, y=7.0.$

The computation results of Fig. 6 show that (1) the assignment factor fa is optimal between 3.0 and 4.0 for three types of charge simulation method, (2) the charge compression method is most effective in the accuracy, and (3) the multiple charges method is better than the basic method when the assignment factor fa is small.

The accuracy of the computation also depends on the density of the contour points. The accuracy can be improved by an increase of the density of the contour points, i. e., the density of the fictitious charges. Fig. 8 shows the basic method with 24 fictitious charges. The following relations are used :

$$a_1 = 1/2 \text{ [m]}, \quad a_2 = d \text{ [m]}, \quad \text{and } fa = 2d. \quad (23)$$

The potential errors ε_b and ε_i are evaluated on the boundary $((0,8) \rightarrow (4, 8) \rightarrow (4, 0), \text{ step length} = 0.25 \text{ [m]})$ and in the space $(0 \leq x \leq 3, 0 \leq y \leq 7, \text{ step length} = 1.0 \text{ [m]})$, respectively. The calculation results are as follows :

- (d) Basic method of $n=24$: $\max |\varepsilon_b| = 0.484 \times 10^{-5}, fa=5.0, x=0.25, y=8.0,$
 $\max |\varepsilon_i| = 0.180 \times 10^{-5}, fa=3.0, x=1.0, y=7.0.$

The basic method of $n=24$ shown in Fig. 8(a) gives a higher accuracy on the boundary and an equivalent accuracy in the space in comparison with the basic method of $n=12$ shown in Fig. 4(a). The basic method of $n=24$ also gives an equivalent accuracy both on the boundary and in the space in comparison with the charge compression method of $n=12$

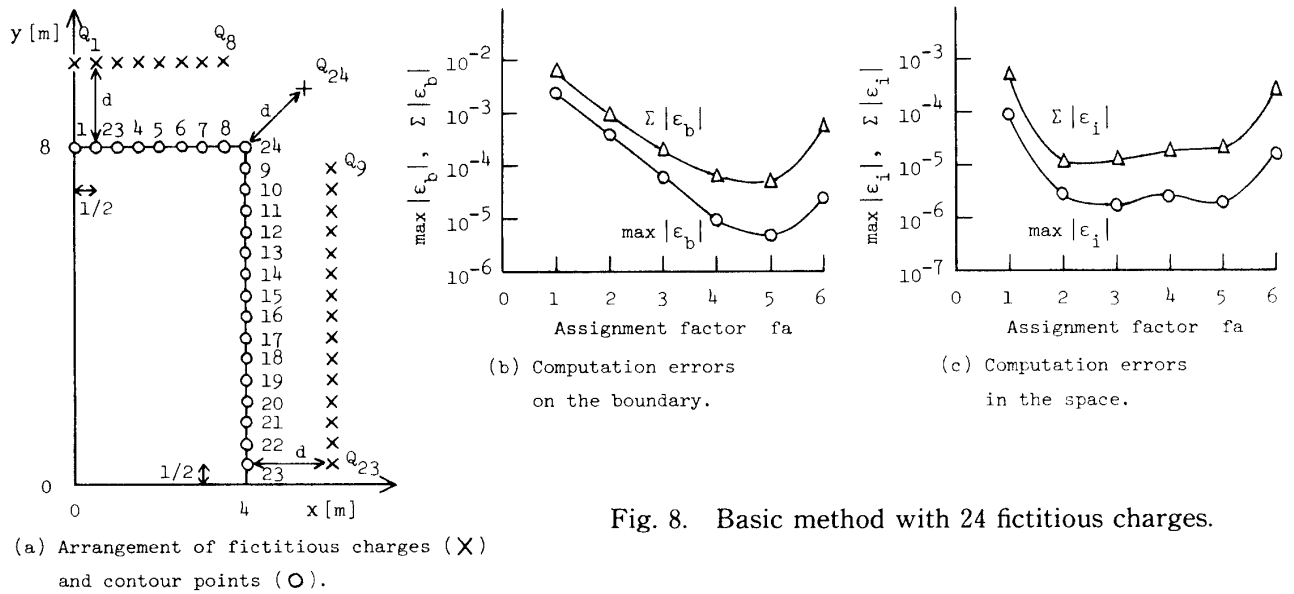


Fig. 8. Basic method with 24 fictitious charges.

shown in Fig. 4 (c). The former method, however, needs a longer computation time in comparison with the latter method, because the computation time for evaluating P_{ij} in Eq. (8), solving Eq. (8), and evaluating ϕ in Eqs. (9) and (10) is increased as the number of fictitious charges n is increased. Therefore the charge compression method is most effective in the accuracy and the computation time among four types of charge simulation method, i. e., Figs. 4 (a), (b), (c) and 8 (a).

Murashima et al. [9] pointed out the property that the maximum absolute value of potential errors in the computation with the charge simulation method exists on the boundary, if the round-off error is negligible. This property is established as shown in Figs. 6 (a), (c) and Figs. 8 (b), (c).

Fig. 7 shows the potential error characteristics on the boundary. It is found that the various errors in Fig. 6 are small when the potential error on the boundary ϵ_b alternates at each contour point as shown in Fig. 7.

All computation in this section are executed on NEAC 3200/50. The values ϕ_{exact} in Eq. (22) are of double precision and the other values are of single precision.

4. Conclusion

Four types of charge simulation method shown in Figs. 4 (a), (b), (c) and 8 (a) have been applied to solve the model problem shown in Fig. 1. The results are summarized as follows :

- (1) The assignment factor f_a is optimal between 3.0 and 5.0 for Figs. 4 (a)–(c) and 8 (a).
- (2) The charge compression method is most effective in the accuracy and the computation time.
- (3) The multiple charges method is better than the basic method for small f_a .
- (4) The maximum absolute value of potential errors exists on the boundary.

The charge simulation method is a suitable way for the solution of many electric practical field problems.

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