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# 剛体球フェルミ粒子系でのリング・ダイアグラム・エネルギーの繰り込み
## －Off Energy Shell Effect－

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# Renormalized Ring Diagram Energy in a Fermion System of Hard Spheres
## －The Off Energy Shell Effect－

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**Abstract**

The off energy shell effect on the contribution to the energy from the renormalized three-particle ring diagram is studied for the ground state of a system of hard sphere fermions. The linked cluster formalism supplemented with infinite partial summation method, such as the $K$-matrix method, fails in giving a finite value to the energy of interest. In place of the $K$-matrix, we make use of the effective potential which improves the pseudopotential introduced by Fermi [1)] and exploited by Lee, Huang and Yang. [2)] Our potential prevents two spheres from penetrating each other. It gives a finite energy correction whose dominant part varies as $a^4 \log a$ with the small value of the core radius $a$. Correction term of order $a^4$ is also presented.

## Introduction

Calculation of the ground state energy of a many-body system of fermions interacting via strong repulsive force (hard core potential) is met with difficulty when one takes account of correlation between three particles. Three-particle ring diagram represents the contribution to the energy from the simplest process in which three particles are involved. Strong repulsion prevents direct application of perturbation method, and one is tempted to renormalize the contribution from the three-particle ring diagram by introducing the $K$-matrix in place of the bare vertex in order to take account of repeated collisions of two excited particles. The collision of two particles in intermediate state is affected by the presence of a third excited particle (off energy shell effect). If one neglects the effect, the contribution to the energy comes out to be of order $(k_0^2 a_s)^3$ in dimensionless scale, where $k_0$ is the Fermi wave number, and the s-wave scattering length $a_s$ is supposed to be a good substitute for the hard core radius $a$. When the off energy shell effect is taken into account, however, the calculation of

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energy is met with ultraviolet divergence.[3) 4)] A naive supposition based on the logarithmic divergence of the energy would be that the energy correction is of order $(k_0 a_s)^4 \log k_0 a_s$. But the supposition contradicts the premise of the $K$-matrix fomalism, because the latter assumes that the energy can be expanded in powers of the strength $g$ of the potential at least in asymptotic sense,[5) 6)] while the expansion is not possible for a term like $(k_0 a_s)^4 \log k_0 a_s$, considering that $a_s$ is analytic at $g = 0$.

This note reports that the divergence difficulty is removed when we use an effective potential for hard core interaction proposed earlier.[7)] The lowest order potential is given by

$$V_1 = \frac{\hbar^2}{ma}\delta(r-a),$$

where $m/2$ is the reduced mass of the interacting pair of particles. We treat $a$ as a smallness parameter. The smallness of the surface of the sphere enables us to treat $V_1$ in perturbational terms. When the wave length $k^{-1}$ is large, the first order wave function of the interacting pair is reduced within the sphere by a factor of order $ka$ from the unperturbed value. In order to improve the wave function, we work out perturbation calculation to the second order in $V_1$ and to the lowest order in an additional potential $V_2$ which is treated as the second order interaction. It is found that the wave function inside the sphere is reduced by a further factor of order $ka$ when we take $V_2 = V_1$. This implies that the part $V_1$ of the infinite hard sphere potential makes the wave function almost vanish inside the sphere. Similarly, of the infinite residual interaction which we leave when we we select $V_1$, the part $V_2$ is mainly responsible for further reduction of the wave function within the sphere. The matrix elements of $V_1$ and $V_2$ sandwiched between two plane wave states $\boldsymbol{q}, \boldsymbol{q}'$ and $\boldsymbol{p}, \boldsymbol{p}'$ are given by

$$(\boldsymbol{q},\boldsymbol{q}'|V_1|\boldsymbol{p},\boldsymbol{p}') = (\boldsymbol{q},\boldsymbol{q}'|V_2|\boldsymbol{p},\boldsymbol{p}') = \frac{4\pi a\hbar^2}{m\Omega} f(\boldsymbol{q}-\boldsymbol{q}')f(\boldsymbol{p}-\boldsymbol{p}')\delta^3(\boldsymbol{q}+\boldsymbol{q}'-\boldsymbol{p}-\boldsymbol{p}'),$$

where $\Omega$ is the volume of the system being considered, and

$$f(\boldsymbol{Q}) = f(Q) = \frac{\sin Qa/2}{Qa/2}. \qquad Q = |\boldsymbol{Q}|$$

The factor $f(\boldsymbol{Q})$ reflects the blocking effect of the strong repulsion on the penetration of particles inside the sphere. It enters in the integrand in the expression of energy as a form factor which prevents the divergence of the integral. In the limit $a \to 0$, $f(\boldsymbol{Q})$ tends to unity. But the approximation $f(\cdots) \approx 1$ is not justified within the integration symbol when the convergence of the integrals over the arguments of $f(\cdots)$ are poor.

## The Off Energy Shell Effect

We are interested in the ground state system of hard sphere fermions of multiplicity

$$\mu = (2S+1)(2T+1),$$

where $S$ and $T$ are the spin and isospin of a fermion. For the sake of later convenience we introduce the symbols

$$w = \mu(\mu-1)(\mu-3)$$

and

$$\theta_q \quad = \quad \theta_q \quad = \quad 1-\overline{\theta}_q \quad = \quad \begin{cases} 1 & |\boldsymbol{q}| = \quad q > k_0 \\ 0. & \qquad\quad q < k_0 \end{cases}$$

The off energy shell effect yields the following correction term to the energy contribution from the renormalized three particle ring diagram

$$E = w\frac{\Omega\hbar^2 a^3}{16m\pi^9}\int d^3\boldsymbol{p_1}\int d^3\boldsymbol{p_2}\int d^3\boldsymbol{p_3}\overline{\theta}_{\boldsymbol{p}_1}\overline{\theta}_{\boldsymbol{p}_2}\overline{\theta}_{\boldsymbol{p}_3}$$
$$\int d^3\boldsymbol{t}\frac{\theta_{\boldsymbol{p}_1+\boldsymbol{t}}\theta_{\boldsymbol{p}_2-\boldsymbol{t}}\theta_{\boldsymbol{p}_3-\boldsymbol{t}}f(\boldsymbol{p_1}-\boldsymbol{p_3}+2\boldsymbol{t})f(\boldsymbol{p_2}-\boldsymbol{p_3}-\boldsymbol{t})f(\boldsymbol{p_2}-\boldsymbol{p_3}+\boldsymbol{t})f(\boldsymbol{p_1}-\boldsymbol{p_2}+2\boldsymbol{t})}{[(\boldsymbol{p_1}+\boldsymbol{t})^2+(\boldsymbol{p_2}-\boldsymbol{t})^2-p_1^2-p_2^2][(\boldsymbol{p_1}+\boldsymbol{t})^2+(\boldsymbol{p_3}-\boldsymbol{t})^2-p_1^2-p_3^2]}$$
$$\left\{1-\frac{a}{\pi^2}\int d^3\boldsymbol{q_0}\frac{\theta_{\boldsymbol{q}_0}\theta_{\boldsymbol{p}_2+\boldsymbol{p}_3-\boldsymbol{t}-\boldsymbol{q}_0}f^2(\boldsymbol{p_2}+\boldsymbol{p_3}-\boldsymbol{t}-2\boldsymbol{q_0})}{(\boldsymbol{p_2}+\boldsymbol{p_3}-\boldsymbol{t}-\boldsymbol{q_0})^2+(\boldsymbol{p_1}+\boldsymbol{t})^2+q_0^2-p_1^2-p_2^2-p_3^2}\right\}.$$

In the last equation we have put $f(\boldsymbol{p_1}-\boldsymbol{p_2}) = 1$. This is permissible because the wave numbers $\boldsymbol{p_1},\boldsymbol{p_2}$ of the particles in the Fermi sea satisfy $p_1, p_2 < k_0 \ll a^{-1}$. The remaining form factors involving $\boldsymbol{t}$ or $\boldsymbol{q_0}$ can not readily be put as unity, although we want to put as unity as many of them as possible. An approximate value $E_0$ of $E$ valid to the leading order in $k_0 a$ is obtained by adopting the following three prescriptions:

(1) ignoring the Pauli exclusion principle which restricts the wave number $\boldsymbol{q_0}$ of an excited particle,

(2) neglecting the hole wave numbers $\boldsymbol{p_1},\boldsymbol{p_2}$, and $\boldsymbol{p_3}$ in the integrand as compared to $\boldsymbol{q_0}$,

(3) neglecting $\boldsymbol{p_1},\boldsymbol{p_2}$, and $\boldsymbol{p_3}$ in estimating the integrand as well as the range of $\boldsymbol{t}$ consistent with the Pauli principle.

Thus

$$E_0 = w\frac{\Omega\hbar^2 a^3}{16m\pi^9}\int d^3\boldsymbol{p_1}\overline{\theta}_{\boldsymbol{p}_1}\int d^3\boldsymbol{p_2}\overline{\theta}_{\boldsymbol{p}_2}\int d^3\boldsymbol{p_3}\overline{\theta}_{\boldsymbol{p}_3}\int d^3\boldsymbol{t}\theta_t\frac{f^2(t)}{2t^2}\frac{f^2(2t)}{2t^2}\left\{1-\frac{a}{\pi^2}\int d^3\boldsymbol{q_0}\frac{f^2(2\boldsymbol{q_0}+\boldsymbol{t})}{(\boldsymbol{t}+\boldsymbol{q_0})^2+q_0^2+t^2}\right\}.$$

In order to correct for the errors introduced by the prescriptions (3),(2) and (1) above, we introduce three terms $E_{1a}$,$E_{1b}$ and $E_2$ to write

$$E = E_0 + E_{1a} + E_{1b} + E_2,$$

where

$$E_{1a} = w\frac{\Omega\hbar^2 a^3}{16m\pi^9}\int d^3\boldsymbol{p_1}\overline{\theta}_{\boldsymbol{p}_1}\int d^3\boldsymbol{p_2}\overline{\theta}_{\boldsymbol{p}_2}\int d^3\boldsymbol{p_3}\overline{\theta}_{\boldsymbol{p}_3}\int d^3\boldsymbol{t}\left\{1-\frac{a}{\pi^2}\int d^3\boldsymbol{q_0}\frac{f^2(2\boldsymbol{q_0}+\boldsymbol{t})}{(\boldsymbol{t}+\boldsymbol{q_0})^2+q_0^2+t^2}\right\}$$
$$\left\{\frac{\theta_{\boldsymbol{p}_1+\boldsymbol{t}}\theta_{\boldsymbol{p}_2-\boldsymbol{t}}\theta_{\boldsymbol{p}_3-\boldsymbol{t}}f(\boldsymbol{p_1}-\boldsymbol{p_3}+2\boldsymbol{t})f(\boldsymbol{p_2}-\boldsymbol{p_3}-\boldsymbol{t})f(\boldsymbol{p_2}-\boldsymbol{p_3}+\boldsymbol{t})f(\boldsymbol{p_1}-\boldsymbol{p_2}+2\boldsymbol{t})}{[(\boldsymbol{p_1}+\boldsymbol{t})^2+(\boldsymbol{p_2}-\boldsymbol{t})^2-p_1^2-p_2^2][(\boldsymbol{p_1}+\boldsymbol{t})^2+(\boldsymbol{p_3}-\boldsymbol{t})^2-p_1^2-p_3^2]}-\theta_t\frac{f^2(t)}{2t^2}\frac{f^2(2t)}{2t^2}\right\},$$

$$E_{1b} = w\frac{\Omega\hbar^2 a^4}{16m\pi^{11}}\int d^3\boldsymbol{p_1}\overline{\theta}_{\boldsymbol{p}_1}\int d^3\boldsymbol{p_2}\overline{\theta}_{\boldsymbol{p}_2}\int d^3\boldsymbol{p_3}\overline{\theta}_{\boldsymbol{p}_3}$$
$$\int d^3\boldsymbol{t}\frac{\theta_{\boldsymbol{p}_1+\boldsymbol{t}}\theta_{\boldsymbol{p}_2-\boldsymbol{t}}\theta_{\boldsymbol{p}_3-\boldsymbol{t}}f(\boldsymbol{p_1}-\boldsymbol{p_3}+2\boldsymbol{t})f(\boldsymbol{p_2}-\boldsymbol{p_3}-\boldsymbol{t})}{(\boldsymbol{p_1}+\boldsymbol{t})^2+(\boldsymbol{p_2}-\boldsymbol{t})^2-p_1^2-p_2^2}\frac{f(\boldsymbol{p_2}-\boldsymbol{p_3}+\boldsymbol{t})f(\boldsymbol{p_1}-\boldsymbol{p_2}+2\boldsymbol{t})}{(\boldsymbol{p_1}+\boldsymbol{t})^2+(\boldsymbol{p_3}-\boldsymbol{t})^2-p_1^2-p_3^2}$$
$$\int d^3\boldsymbol{q_0}\left\{\frac{f^2(2\boldsymbol{q_0}+\boldsymbol{t})}{(\boldsymbol{t}+\boldsymbol{q_0})^2+q_0^2+t^2}-\frac{f^2(\boldsymbol{p_2}+\boldsymbol{p_3}-\boldsymbol{t}-2\boldsymbol{q_0})}{(\boldsymbol{p_2}+\boldsymbol{p_3}-\boldsymbol{t}-\boldsymbol{q_0})^2+(\boldsymbol{p_1}+\boldsymbol{t})^2+q_0^2-p_1^2-p_2^2-p_3^2}\right\}$$

and

$$E_2 = w\frac{\Omega\hbar^2 a^4}{16m\pi^{11}}\int d^3\boldsymbol{p_1}\overline{\theta}_{\boldsymbol{p}_1}\int d^3\boldsymbol{p_2}\overline{\theta}_{\boldsymbol{p}_2}\int d^3\boldsymbol{p_3}\overline{\theta}_{\boldsymbol{p}_3}$$
$$\int d^3\boldsymbol{t}\int d^3\boldsymbol{q_0}\frac{\theta_{\boldsymbol{p}_1+\boldsymbol{t}}\theta_{\boldsymbol{p}_2-\boldsymbol{t}}\theta_{\boldsymbol{p}_3-\boldsymbol{t}}(1-\theta_{\boldsymbol{q}_0}\theta_{\boldsymbol{p}_2+\boldsymbol{p}_3-\boldsymbol{t}-\boldsymbol{q}_0})}{(\boldsymbol{p_2}+\boldsymbol{p_3}-\boldsymbol{t}-\boldsymbol{q_0})^2+(\boldsymbol{p_1}+\boldsymbol{t})^2+q_0^2-p_1^2-p_2^2-p_3^2}$$
$$\frac{f^2(\boldsymbol{p_2}+\boldsymbol{p_3}-\boldsymbol{t}-2\boldsymbol{q_0})f(\boldsymbol{p_1}-\boldsymbol{p_3}+2\boldsymbol{t})f(\boldsymbol{p_2}-\boldsymbol{p_3}-\boldsymbol{t})f(\boldsymbol{p_2}-\boldsymbol{p_3}+\boldsymbol{t})f(\boldsymbol{p_1}-\boldsymbol{p_2}+2\boldsymbol{t})}{[(\boldsymbol{p_1}+\boldsymbol{t})^2+(\boldsymbol{p_2}-\boldsymbol{t})^2-p_1^2-p_2^2][(\boldsymbol{p_1}+\boldsymbol{t})^2+(\boldsymbol{p_3}-\boldsymbol{t})^2-p_1^2-p_3^2]}.$$

We aim at estimating $E$ correctly to terms of order $(k_0a)^4$ in dimensionless scale. The following formula helps us reduce $E_0$ and $E_{1a}$,

$$1-\frac{a}{\pi^2}\int d^3\boldsymbol{q_0}\frac{f^2(2\boldsymbol{q_0}+\boldsymbol{t})}{(\boldsymbol{t}+\boldsymbol{q_0})^2+q_0^2+t^2} = 1-\frac{1}{\sqrt{3}at}(1-e^{-\sqrt{3}at}).$$

In paticular, $E_0$ can be greatly simplified to

$$E_0 = w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2k_0^2}{2m}\frac{16}{9\pi^3}k_0^4a^3\int_{k_0}^{\infty}\frac{dt}{t^2}\left(\frac{\sin at}{at}\frac{\sin at/2}{at/2}\right)^2\left(1-\frac{1}{\sqrt{3}at}(1-e^{-\sqrt{3}at})\right).$$

The last factor in the integrand of the above expression of $E_0$ can not be treated by power expansion in terms of $at$ for our present purpose. But the same factor entering in the integrand of $E_{1a}$ can be approximated by

$$1-\frac{1}{\sqrt{3}at}(1-e^{-\sqrt{3}at}) \approx \frac{\sqrt{3}at}{2}.$$

Similarly we can use the following approximation in estimating $E_{1b}$,

$$\int d^3q_0 \left\{ \frac{f^2(2q_0+t)}{(t+q_0)^2+q_0^2+t^2} - \frac{f^2(p_2+p_3-t-2q_0)}{(p_2+p_3-t-q_0)^2+(p_1+t)^2+q_0^2-p_1^2-p_2^2-p_3^2} \right\}$$
$$\approx \begin{cases} \pi^2(\sqrt{R(t,p_1,p_2,p_3)}-\frac{\sqrt{3}}{2}t), & R(t,p_1,p_2,p_3)>0 \\ -\pi^2\frac{\sqrt{3}}{2}t, & R(t,p_1,p_2,p_3)<0 \end{cases}$$

where

$$R(t,p_1,p_2,p_3)=\frac{1}{4}(p_2+p_3-t)^2+\frac{1}{2}[(p_1+t)^2-p_1^2-p_2^2-p_3^2].$$

The remaining factors $f(\cdots)$ in $E_{1a}$ and $E_{1b}$ can be approximated by unity. We put the two resultant expressions of $E_{1a}$ and $E_{1b}$ together to write

$$E_1 = E_{1a}+E_{1b}$$
$$= w\frac{\Omega\hbar^2a^4}{16m\pi^9}\int d^3p_1\bar{\theta}_{p_1}\int d^3p_2\bar{\theta}_{p_2}\int d^3p_3\bar{\theta}_{p_3}\int d^3t$$
$$\left\{ \frac{\theta_{p_1+t}\theta_{p_2-t}\theta_{p_3-t}\sqrt{R(t,p_1,p_2,p_3)}}{[(p_1+t)^2+(p_2-t)^2-p_1^2-p_2^2][(p_1+t)^2+(p_3-t)^2-p_1^2-p_3^2]} - \theta_t\frac{\sqrt{3}}{8}\frac{1}{t^3} \right\},$$

where the integration variables are subject to the condition

$$R(t,p_1,p_2,p_3)>0.$$

In estimating the correction term $E_2$, we can put all form factors as unity. This is because the range of the vector $q_0$ is bounded owing to the factor $(1-\theta_{q_0}\theta_{p_2+p_3-t-q_0})$ and that the energy denominator is of order $\max(t^6, t^4q_0^2)$ for large values of $t$ and $q_0$. Carrying out the integration over $q_0$ we find

$$E_2 = w\frac{\Omega\hbar^2a^4}{16\pi^{10}m}\int d^3p_1\bar{\theta}_{p_1}\int d^3p_2\bar{\theta}_{p_2}\int d^3p_3\bar{\theta}_{p_3}$$
$$\int d^3t\frac{\theta_{p_1+t}\theta_{p_2-t}\theta_{p_3-t}\ \Xi(\ |p_2+p_3-t|,\ R(t,p_1,p_2,p_3)\ )}{[(p_1+t)^2+(p_2-t)^2-p_1^2-p_2^2][(p_1+t)^2+(p_3-t)^2-p_1^2-p_3^2]},$$

where $\Xi(|s|,R)$ is defined by

$$\Xi(|s|,R)=\frac{1}{2\pi}\int d^3q\mathrm{H.W.}\frac{1-\theta_{q+s/2}\theta_{q-s/2}}{q^2+R},$$

in which the principal part integral is implied if $R<0$. The concrete expression of $\Xi(|s|,R)$ can be given, using the symbols $\kappa_0=k_0-s/2$ and $\kappa_2=k_0+s/2$ , as follows

$$\Xi(s,R)=\begin{cases} \frac{\kappa_0\kappa_2+R}{s}\log\frac{\kappa_2^2+R}{\kappa_0\kappa_2+R}-2\sqrt{R}\arctan\frac{\kappa_2}{\sqrt{R}}+\kappa_2, & (R>0,\ k_0>s/2) \\ \frac{\kappa_0\kappa_2+R}{s}\log\frac{\kappa_2^2+R}{\kappa_0^2+R}-2\sqrt{R}\arctan\frac{2k_0\sqrt{R}}{R-\kappa_0\kappa_2}+2k_0, & (R>0,\ k_0<s/2) \\ \frac{\kappa_0\kappa_2+R}{s}\log\left|\frac{\kappa_2^2+R}{\kappa_0\kappa_2+R}\right|+\sqrt{-R}\log\left|\frac{\kappa_2-\sqrt{-R}}{\kappa_2+\sqrt{-R}}\right|+\kappa_2, & (R<0,\ k_0>s/2) \\ \text{We need not consider the case.} & (R<0,\ k_0<s/2) \end{cases}$$

Let us now turn to numerical results. Standard calculus gives the following expression for $E_0$ valid to order $(k_0a)^4$ in unit of energy $\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}$

$$E_0 = w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}(k_0a)^4(\frac{2}{\sqrt{3}\pi})^3\Big\{ -\log k_0a$$
$$-\gamma - \sqrt{3}\Big(\frac{59}{360}\pi + \frac{1}{6}\arctan\frac{\sqrt{3}}{2}\Big) + \frac{49}{20} - \frac{104}{135}\log 2 - \frac{3}{4}\log 3 + \frac{143}{2160}\log 7\Big\} + \cdots$$
$$= w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}(k_0a)^4\left(-\left(\frac{2}{\sqrt{3}\pi}\right)^3\log k_0a - 0.022550\right),$$

where $\gamma = 0.577215\cdots$ is Euler's constant.

For numerical estimation of the integral $E_1$, we find it convenient to use as variables of integration the absolute values of $\boldsymbol{s} = \boldsymbol{p_2} + \boldsymbol{p_3} - \boldsymbol{t}$, $\boldsymbol{q_2} = \boldsymbol{p_2} - \boldsymbol{t}$ and the component $p_{1||}$ of $\boldsymbol{p_1}$ parallel to $\boldsymbol{t}$. We also make use of the relation

$$q_3^2 = (\boldsymbol{p_3} - \boldsymbol{t})^2 = \frac{1}{4q_2^2}\Big\{(t^2 + s^2 - p_2^2 - p_3^2)^2 + A_s^2 + A_p^2 - 2A_sA_p\cos\alpha\Big\},$$

where

$$A_s = \sqrt{s^2 - (p_3 - q_2)^2}\sqrt{(p_3 + q_2)^2 - s^2},$$
$$A_p = \sqrt{t^2 - (p_2 - q_2)^2}\sqrt{(p_2 + q_2)^2 - t^2},$$

and $\alpha$ is the angle between two planes, one determined by $\boldsymbol{q_2}$ and $\boldsymbol{p_2}$ and another determind by $\boldsymbol{q_2}$ and $\boldsymbol{p_3}$. Thus

$$E_1 = w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}(k_0a)^4\frac{6}{\pi^4}\int_0^\infty dt\Big\{-\theta_t\bar{\theta}_{t-3k_0}\frac{4\sqrt{3}\pi}{27t}$$
$$+\frac{t}{k_0^9}\int_{max(0,k_0-t)}^{k_0} dp_2p_2\int_{\max(k_0,t-p_2)}^{t+p_2} dq_2\int_{L_3}^{k_0} dp_3p_3\int_{L_s}^{H_s} dss\int_{L_\alpha}^{\pi} d\alpha\int_{L_{1||}}^{k_0} dp_{1||}$$
$$\min[t(t+2p_{1||}), k_0^2 - p_{1||}^2]\left(-\theta_{t-3k_0}\frac{\sqrt{3}}{4t^3} + \frac{\sqrt{s^2 + 2(t^2 + 2tp_{1||} - p_2^2 - p_3^2)}}{(t^2 + 2tp_{1||} + q_2^2 - p_2^2)(t^2 + 2tp_{1||} + q_3^2 - p_3^2)}\right)\Big\}.$$

The upper and lower bounds of integration in the last equation are complicated due to the combined effect of the Pauli principle and the requirement $R(\boldsymbol{t}, \boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}) > 0$. Each of them can depend on other variables of integration to be carried out after the quadrature being considered is performed. The lower bound $L_3$ of $p_3$ is given by

$$L_3 = \begin{cases} 0, & t > k_0 \\ k_0 - t, & t < k_0 \quad \text{and} \quad q_2^2 > p_2^2(1 + t/k_0) + k_0t - 7t^2 \\ c, & \text{otherwise} \end{cases}$$

where

$$c^2 = \frac{1}{4}(3k_0^2 + 4k_0t + 4t^2 - p_2^2 + \frac{1}{2t^2 + 2q_2^2 - p_2^2}\{(k_0^2 + p_2^2 - 4k_0t)(p_2^2 - 4t^2) - 2A_p\sqrt{\eta}\}),$$

and

$$\eta = 8k_0^2t(k_0 - t) + k_0^2(8q_2^2 - k_0^2 - 7p_2^2) + p_2^2(k_0^2 - p_2^2 + 8k_0t).$$

The lower bound $L_s$ and the upper bound $H_s$ of $s$ are given by

$$L_s = \begin{cases} q_2 - p_3, & t > k_0 \quad \text{and} \quad p_3 < -b \\ u_-, & t > k_0 \quad \text{and} \quad p_3 > -b \\ u_-, & t < k_0 \quad \text{and} \quad p_3 < c \\ \sqrt{2(p_3^2 + p_2^2 - t^2 - 2k_0t)}, & t < k_0 \quad \text{and} \quad p_3 > c \end{cases}$$

and

$$H_s = \begin{cases} q_2 + p_3, & p_3 > b \\ u_+, & \text{otherwise} \end{cases}$$

where

$$b = \frac{1}{2q_2}\left\{\sqrt{4(k_0^2 - p_2^2)q_2^2 + (p_2^2 + q_2^2 - t^2)^2} - (t^2 + q_2^2 - p_2^2)\right\},$$

$$u_\pm^2 = \frac{1}{4t^2}\left\{(q_2^2 + k_0^2 - p_2^2 - p_3^2)^2 + (A_k \pm A_p)^2\right\},$$

and

$$A_k = \sqrt{k_0^2 - (p_3 - t)^2}\sqrt{(p_3 + t)^2 - k_0^2}.$$

The lower bounds $L_\alpha$ and $L_{1||}$ of $\alpha$ and $p_{1||}$ respectively can be written as

$$L_\alpha = \arccos\min(Z, 1),$$

in which

$$Z = \frac{1}{2A_sA_p}\left(A_s^2 + A_p^2 + (s^2 + t^2 - p_2^2 - p_3^2) - 4k_0^2q_2^2\right)$$

and

$$L_{1||} = \frac{1}{2t}\max(-2k_0t, -t^2, p_2^2 + p_3^2 - t^2 - s^2/2).$$

With these bounds the integral $E_1$ turns out to be

$$E_1 = w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2k_0^2}{2m}(k_0a)^4\left(-\left(\frac{2}{\sqrt{3}\pi}\right)^3\log k_0a - 0.0225504\right).$$

The remaining integral $E_2$ can be transformed as

$$E_2 = w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}(k_0 a)^4 \frac{12}{\pi^5 k_0^9}\int_0^\infty dt \int_{\max(0,k_0-t)}^{k_0} dp_{1||}\min[t(t+2p_{1||}),k_0^2-p_{1||}^2]$$
$$\int_{max(0,k_0-t)}^{k_0} dp_2 p_2 \int_{\max(k_0,t-p_2)}^{t+p_2} dq_2 q_2 \frac{1}{t^2+2tp_{1||}+q_2^2-p_2^2}\int_{max(0,k_0-t)}^{k_0} dp_3 p_3$$
$$\int_{\max(k_0,t-p_3)}^{t+p_3} dq_3 q_3 \frac{1}{t^2+2tp_{1||}+q_3^2-p_3^2}\int_0^\pi d\phi\ \Xi\Big(s,\ \tfrac{1}{4}s^2+\tfrac{1}{2}(t^2+2tp_{1||}-p_2^2-p_3^2)\Big),$$

where

$$s = |\boldsymbol{p_2}+\boldsymbol{p_3}-\boldsymbol{t}| = \frac{1}{2t}\sqrt{(q_2^2+q_3^2-p_2^2-p_3^2)^2+A_p^2+A_q^2-2A_pA_q\cos\phi},$$
$$A_q = \sqrt{(q_3+p_3)^2-t^2}\sqrt{t^2-(q_3-p_3)^2},$$

and $\phi$ is the angle between two directions determined by $\boldsymbol{t}\times\boldsymbol{p_2}$ and $\boldsymbol{t}\times\boldsymbol{p_3}$. Our numerical result is

$$E_2 = w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}(k_0 a)^4(-0.000997).$$

**Conclusion**

Putting the above results together, we find the correlation energy due to the off energy shell effect for the renormalized three-particle ring diagram as

$$E = w\frac{\Omega k_0^3}{6\pi^2}\frac{\hbar^2 k_0^2}{2m}(k_0 a)^4(-0.0496545\log k_0 a - 0.03246).$$

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